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THE INTEGRATION OF FUNCTIONS OF A SINGLE VARIABLE

by

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PREFACE

THIS tract has been long out of print, and there is still some demand for it. I did not publish a second edition before, because I intended to incorporate its contents in a larger treatise on the subject which I had arranged to write in collaboration with Dr Bromwich. Four or five years have passed, and it seems very doubtful whether either of us will ever find the time to carry out our intention. I have therefore decided to republish the tract.

The new edition differs from the first in one important point only. In the first edition I reproduced a proof of Abel's which Mr J. E. Littlewood afterwards discovered to be invalid. The correction of this error has led me to rewrite a few sections (pp. 36–41 of the present edition) completely. The proof which I give now is due to Mr H. T. J. Norton. I am also indebted to Mr Norton, and to Mr S. Pollard, for many other criticisms of a less important character.

G. H. H.

January 1916.

CONTENTS

THE INTEGRATION OF FUNCTIONS OF A SINGLE VARIABLE

I. Introduction

The problem considered in the following pages is what is sometimes called the problem of 'indefinite integration' or of 'finding a function whose differential coefficient is a given function'. These descriptions are vague and in some ways misleading; and it is necessary to define our problem more precisely before we proceed further.

Let us suppose for the moment that $f(x)$ is a real continuous function of the real variable x. We wish to determine a function y whose differential coefficient is $f(x)$, or to solve the equation

$$\frac{dy}{dx} = f(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

A little reflection shows that this problem may be analysed into a number of parts.

We wish, first, to know whether such a function as y necessarily exists, whether the equation (1) has always a solution; whether the solution, if it exists, is unique; and what relations hold between different solutions, if there are more than one. The answers to these questions are contained in that part of the theory of functions of a real variable which deals with 'definite integrals'. The definite integral

$$y = \int_a^x f(t)\,dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

which is defined as the limit of a certain sum, is a solution of the equation (1). Further

$$y + C \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where C is an arbitrary constant, is also a solution, and all solutions of (1) are of the form (3).

These results we shall take for granted. The questions with which we shall be concerned are of a quite different character. They are questions as to the functional form of y when $f(x)$ is a function of some stated form. It is sometimes said that the problem of indefinite integration is that of 'finding an actual expression for y when $f(x)$ is given'. This statement is however still lacking in precision. The theory of definite integrals provides us not only with a proof of the existence of a solution, but also with an expression for it, an expression in the form of a limit. The problem of indefinite integration can be stated precisely only when we introduce sweeping restrictions as to the classes of functions and the modes of expression which we are considering.

Let us suppose that $f(x)$ belongs to some special class of functions $\mathfrak{F}$. Then we may ask whether y is itself a member of $\mathfrak{F}$, or can be expressed, according to some simple standard mode of expression, in terms of functions which are members of $\mathfrak{F}$. To take a trivial example, we might suppose that $\mathfrak{F}$ is the class of polynomials with rational coefficients; the answer would then be that y is in all cases itself a member of $\mathfrak{F}$.

The range and difficulty of our problem will depend upon our choice of (1) a class of functions and (2) a standard 'mode of expression'. We shall, for the purposes of this tract, take $\mathfrak{F}$ to be the class of *elementary functions*, a class which will be defined precisely in the next section, and our mode of expression to be that of *explicit expression in finite terms, i.e.* by formulae which do not involve passages to a limit.

One or two more preliminary remarks are needed. The subject-matter of the tract forms a chapter in the 'integral calculus'*, but does not depend in any way on any direct theory of integration. Such an equation as

$$y = \int f(x)\,dx \quad\text{..............................}(4)$$

is to be regarded as merely another way of writing (1): the integral sign is used merely on grounds of technical convenience, and might be eliminated throughout without any substantial change in the argument.

* Euler, the first systematic writer on the 'integral calculus', defined it in a manner which identifies it with the theory of differential equations: 'calculus integralis est methodus, ex data differentialium relatione inveniendi relationem ipsarum quantitatum' (*Institutiones calculi integralis*, p. 1). We are concerned only with the special equation (1), but all the remarks we have made may be generalised so as to apply to the wider theory.

The variable x is in general supposed to be complex. But the tract should be intelligible to a reader who is not acquainted with the theory of analytic functions and who regards x as real and the functions of x which occur as real or complex functions of a real variable.

The functions with which we shall be dealing will always be such as are regular except for certain special values of x. These values of x we shall simply ignore. The meaning of such an equation as

$$\int \frac{dx}{x} = \log x$$

is in no way affected by the fact that $1/x$ and $\log x$ have infinities for $x = 0$.

II. Elementary functions and their classification

An *elementary function* is a member of the class of functions which comprises

(i) rational functions,

(ii) algebraical functions, explicit or implicit,

(iii) the exponential function e^x,

(iv) the logarithmic function $\log x$,

(v) all functions which can be defined by means of any finite combination of the symbols proper to the preceding four classes of functions.

A few remarks and examples may help to elucidate this definition.

1. A *rational function* is a function defined by means of any finite combination of the elementary operations of addition, multiplication, and division, operating on the variable x.

It is shown in elementary algebra that any rational function of x may be expressed in the form

$$f(x) = \frac{a_0 x^m + a_1 x^{m-1} + \dots + a_m}{b_0 x^n + b_1 x^{n-1} + \dots + b_n},$$

where m and n are positive integers, the a's and b's are constants, and the numerator and denominator have no common factor. We shall adopt this expression as the standard form of a rational function. It is hardly necessary to remark that it is in no way involved in the definition of a rational function that these constants should be rational or algebraical* or real *numbers*. Thus

$$\frac{x^2 + x + i\sqrt{2}}{x\sqrt{2} - e}$$

is a rational function.

2. An *explicit algebraical function* is a function defined by means of any finite combination of the four elementary operations and any finite number of operations of root extraction. Thus

$$\frac{\sqrt{(1+x)} - \sqrt[3]{(1-x)}}{\sqrt{(1+x)} + \sqrt[3]{(1-x)}}, \quad \sqrt{\{x + \sqrt{(x + \sqrt{x})}\}}, \quad \left(\frac{x^2 + x + i\sqrt{2}}{x\sqrt{2} - e}\right)^{\frac{2}{3}}$$

are explicit algebraical functions. And so is $x^{m/n}$ (*i.e.* $\sqrt[n]{x^m}$) for any integral values of m and n. On the other hand

$$x^{\sqrt{2}}, \quad x^{1+i}$$

are not algebraical functions at all, but transcendental functions, as irrational or complex powers are defined by the aid of exponentials and logarithms.

Any explicit algebraical function of x satisfies an equation

$$P_0 y^n + P_1 y^{n-1} + \dots + P_n = 0$$

whose coefficients are polynomials in x. Thus, for example, the function

$$y = \sqrt{x} + \sqrt{(x + \sqrt{x})}$$

satisfies the equation

$$y^4 - (4y^2 + 4y + 1)\,x = 0.$$

The converse is not true, since it has been proved that in general equations of degree higher than the fourth have no roots which are explicit algebraical functions of their coefficients. A simple example is given by the equation

$$y^5 - y - x = 0.$$

We are thus led to consider a more general class of functions, *implicit* algebraical functions, which includes the class of explicit algebraical functions.

* An algebraical number is a number which is the root of an algebraical equation whose coefficients are integral. It is known that there are numbers (such as e and π) which are not roots of any such equation. See, for example, Hobson's *Squaring the circle* (Cambridge, 1913).

3. An *algebraical function* of x is a function which satisfies an equation

$$P_0 y^n + P_1 y^{n-1} + \dots + P_n = 0 \quad \dots\dots\dots\dots\dots\dots(1)$$

whose coefficients are polynomials in x.

Let us denote by $P(x, y)$ a polynomial such as occurs on the left-hand side of (1). Then there are two possibilities as regards any particular polynomial $P(x, y)$. Either it is possible to express $P(x, y)$ as the product of two polynomials of the same type, neither of which is a mere constant, or it is not. In the first case $P(x, y)$ is said to be *reducible*, in the second *irreducible*. Thus

$$y^4 - x^2 = (y^2 + x)(y^2 - x)$$

is reducible, while both $y^2 + x$ and $y^2 - x$ are irreducible.

The equation (1) is said to be reducible or irreducible according as its left-hand side is reducible or irreducible. A reducible equation can always be replaced by the logical alternative of a number of irreducible equations. Reducible equations are therefore of subsidiary importance only; and we shall always suppose that the equation (1) is irreducible.

An algebraical function of x is regular except at a finite number of points which are *poles* or *branch points* of the function. Let D be any closed simply connected domain in the plane of x which does not include any branch point. Then there are n and only n distinct functions which are one-valued in D and satisfy the equation (1). These n functions will be called the *roots* of (1) in D. Thus if we write

$$x = r(\cos\theta + i\sin\theta),$$

where $-\pi < \theta \leqslant \pi$, then the roots of

$$y^2 - x = 0,$$

in the domain

$$0 < r_1 \leqslant r \leqslant r_2, \quad -\pi < -\pi + \delta \leqslant \theta \leqslant \pi - \delta < \pi,$$

are $\sqrt{x}$ and $-\sqrt{x}$, where

$$\sqrt{x} = \sqrt{r}(\cos\tfrac{1}{2}\theta + i\sin\tfrac{1}{2}\theta).$$

The relations which hold between the different roots of (1) are of the greatest importance in the theory of functions*. For our present purposes we require only the two which follow.

(i) Any symmetric polynomial in the roots $y_1, y_2, \dots, y_n$ of (1) is a rational function of x.

* For fuller information the reader may be referred to Appell and Goursat's *Théorie des fonctions algébriques*.

(ii) Any symmetric polynomial in $y_2, y_3, \dots, y_n$ is a polynomial in y_1 with coefficients which are rational functions of x.

The first proposition follows directly from the equations

$$\Sigma y_1 y_2 \dots y_s = (-1)^s (P_{n-s}/P_0) \qquad (s=1, 2, \dots, n).$$

To prove the second we observe that

$$\sum_{2,3,\dots} y_2 y_3 \dots y_s = \sum_{1,2,\dots} y_1 y_2 \dots y_{s-1} - y_1 \sum_{2,3,\dots} y_2 y_3 \dots y_{s-1},$$

so that the theorem is true for $\Sigma y_2 y_3 \dots y_s$ if it is true for $\Sigma y_2 y_3 \dots y_{s-1}$. It is certainly true for

$$y_2 + y_3 + \dots + y_n = (y_1 + y_2 + \dots + y_n) - y_1.$$

It is therefore true for $\Sigma y_2 y_3 \dots y_s$, and so for any symmetric polynomial in $y_2, y_3, \dots, y_n$.

4. Elementary functions which are not rational or algebraical are called *elementary transcendental functions* or elementary transcendents. They include all the remaining functions which are of ordinary occurrence in elementary analysis.

The trigonometrical (or circular) and hyperbolic functions, direct and inverse, may all be expressed in terms of exponential or logarithmic functions by means of the ordinary formulae of elementary trigonometry. Thus, for example,

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}, \qquad \sinh x = \frac{e^x - e^{-x}}{2},$$

$$\text{arc} \tan x = \frac{1}{2i} \log\left(\frac{1+ix}{1-ix}\right), \quad \text{arg} \tanh x = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right).$$

There was therefore no need to specify them particularly in our definition.

The elementary transcendents have been further classified in a manner first indicated by Liouville*. According to him a function is a transcendent *of the first order* if the signs of exponentiation or of the taking of logarithms which occur in the formula which defines it apply only to rational or algebraical functions. For example

$$xe^{-x^2}, \; e^{x^2} + e^x \surd(\log x)$$

are of the first order; and so is

$$\text{arc} \tan \frac{y}{\surd(1+x^2)},$$

* 'Mémoire sur la classification des transcendantes, et sur l'impossibilité d'exprimer les racines de certaines équations en fonction finie explicite des coefficients', *Journal de mathématiques*, ser. 1, vol. 2, 1837, pp. 56–104; 'Suite du mémoire...', *ibid.* vol. 3, 1838, pp. 523–546.

where y is defined by the equation

$$y^5 - y - x = 0;$$

and so is the function y defined by the equation

$$y^5 - y - e^x \log x = 0.$$

An elementary transcendent *of the second order* is one defined by a formula in which the exponentiations and takings of logarithms are applied to rational or algebraical functions or to transcendents of the first order. This class of functions includes many of great interest and importance, of which the simplest are

$$e^{e^x}, \log \log x.$$

It also includes irrational and complex powers of x, since, *e.g.*,

$$x^{\sqrt{2}} = e^{\sqrt{2} \log x}, \quad x^{1+i} = e^{(1+i)\log x};$$

the function

$$x^x = e^{x \log x};$$

and the logarithms of the circular functions.

It is of course presupposed in the definition of a transcendent of the second kind that the function in question is incapable of expression as one of the first kind or as a rational or algebraical function. The function

$$e^{\log R(x)},$$

where $R(x)$ is rational, is not a transcendent of the second kind, since it can be expressed in the simpler form $R(x)$.

It is obvious that we can in this way proceed to define transcendents of the nth order for all values of n. Thus

$$\log \log \log x, \ \log \log \log \log x, \ \ldots\ldots$$

are of the third, fourth, ... orders.

Of course a similar classification of algebraical functions can be and has been made. Thus we may say that

$$\sqrt{x}, \quad \sqrt{(x + \sqrt{x})}, \quad \sqrt{\{x + \sqrt{(x + \sqrt{x})}\}}, \ \ldots\ldots$$

are algebraical functions of the first, second, third, ... orders. But the fact that there is a general theory of algebraical equations and therefore of *implicit* algebraical functions has deprived this classification of most of its importance. There is no such general theory of elementary transcendental equations*, and therefore we shall not

* The natural generalisations of the theory of algebraical equations are to be found in parts of the theory of differential equations. See Königsberger, 'Bemerkungen zu Liouville's Classificirung der Transcendenten', *Math. Annalen*, vol. 28, 1886, pp. 483–492.

rank as 'elementary' functions defined by transcendental equations such as

$$y = x \log y,$$

but incapable (as Liouville has shown that in this case y is incapable) of explicit expression in finite terms.

5. The preceding analysis of elementary transcendental functions rests on the following theorems:

(*a*) e^x is not an algebraical function of x;

(*b*) $\log x$ is not an algebraical function of x;

(*c*) $\log x$ is not expressible in finite terms by means of signs of exponentiation and of algebraical operations, explicit or implicit*;

(*d*) transcendental functions of the first, second, third, ... orders actually exist.

A proof of the first two theorems will be given later, but limitations of space will prevent us from giving detailed proofs of the third and fourth. Liouville has given interesting extensions of some of these theorems: he has proved, for example, that no equation of the form

$$Ae^{\alpha p} + Be^{\beta p} + \dots + Re^{\rho p} = S,$$

where $p, A, B, \dots, R, S$ are algebraical functions of x, and $\alpha, \beta, \dots, \rho$ different constants, can hold for all values of x.

III. The integration of elementary functions. Summary of results

In the following pages we shall be concerned exclusively with the problem of the integration of elementary functions. We shall endeavour to give as complete an account as the space at our disposal permits of the progress which has been made by mathematicians towards the solution of the two following problems:

(i) *if $f(x)$ is an elementary function, how can we determine whether its integral is also an elementary function?*

(ii) *if the integral is an elementary function, how can we find it?*

It would be unreasonable to expect complete answers to these questions. But sufficient has been done to give us a tolerably complete insight into the nature of the answers, and to ensure that it

* For example, $\log x$ cannot be equal to e^y, where y is an algebraical function of x.

shall not be difficult to find the complete answers in any particular case which is at all likely to occur in elementary analysis or in its applications.

It will probably be well for us at this point to summarise the principal results which have been obtained.

1. The integral of a rational function (IV.) is *always* an elementary function. It is either rational or the sum of a rational function and of a finite number of constant multiples of logarithms of rational functions (IV., 1).

If certain constants which are the roots of an algebraical equation are treated as known then the form of the integral can always be determined completely. But as the roots of such equations are not in general capable of explicit expression in finite terms, it is not in general possible to express the integral in an absolutely explicit form (IV.; 2, 3).

We can always determine, by means of a finite number of the elementary operations of addition, multiplication, and division, whether the integral is rational or not. If it is rational, we can determine it completely by means of such operations; if not, we can determine its rational part (IV.; 4, 5).

The solution of the problem in the case of rational functions may therefore be said to be complete; for the difficulty with regard to the explicit solution of algebraical equations is one not of inadequate knowledge but of proved impossibility (IV., 6).

2. The integral of an algebraical function (V.), explicit or implicit, may or may not be elementary.

If y is an algebraical function of x then the integral $\int y\,dx$, or, more generally, the integral

$$\int R(x, y)\,dx,$$

where R denotes a rational function, is, if an elementary function, either algebraical or the sum of an algebraical function and of a finite number of constant multiples of logarithms of algebraical functions. All algebraical functions which occur in the integral are *rational functions of x and y* (V.; 11–14, 18).

These theorems give a precise statement of a general principle enunciated by Laplace*: '*l'intégrale d'une fonction différentielle*

* *Théorie analytique des probabilités*, p. 7.

(algébrique) ne peut contenir d'autres quantités radicaux que celles qui entrent dans cette fonction'; and, we may add, cannot contain *exponentials* at all. Thus it is impossible that

$$\int \frac{dx}{\sqrt{(1+x^2)}}$$

should contain e^x or $\sqrt{(1-x)}$: the appearance of these functions in the integral could only be apparent, and they could be eliminated before differentiation. Laplace's principle really rests on the fact, of which it is easy enough to convince oneself by a little reflection and the consideration of a few particular cases (though to give a rigorous proof is of course quite another matter), that *differentiation will not eliminate exponentials or algebraical irrationalities.* Nor, we may add, will it eliminate logarithms except when they occur in the simple form

$$A \log \phi(x),$$

where A is a constant, and this is why logarithms can only occur in this form in the integrals of rational or algebraical functions.

We have thus a general knowledge of the form of the integral of an algebraical function y, when it is itself an elementary function. Whether this is so or not of course depends on the nature of the equation $f(x, y) = 0$ which defines y. If this equation, when interpreted as that of a curve in the plane (x, y), represents a *unicursal* curve, *i.e.* a curve which has the maximum number of double points possible for a curve of its degree, or whose *deficiency* is zero, then x and y can be expressed simultaneously as rational functions of a third variable t, and the integral can be reduced by a substitution to that of a rational function (v.; 2, 7–9). In this case, therefore, the integral is always an elementary function. But this condition, though sufficient, is not necessary. It is in general true that, when $f(x, y) = 0$ is not unicursal, the integral is not an elementary function but a new transcendent; and we are able to classify these transcendents according to the deficiency of the curve. If, for example, the deficiency is unity, then the integral is in general a transcendent of the kind known as *elliptic integrals*, whose characteristic is that they can be transformed into integrals containing no other irrationality than the square root of a polynomial of the third or fourth degree (v., 20). But there are infinitely many cases in which the integral can be expressed by algebraical functions and logarithms. Similarly there are infinitely many cases in which integrals associated with curves whose deficiency is greater than unity are in reality reducible to elliptic integrals. Such abnormal cases have formed the subject of many exceedingly interesting researches, but no general method has been devised by which we can always tell, after a finite series of operations, whether any given integral is really elementary, or elliptic, or belongs to a higher order of transcendents.

When $f(x, y) = 0$ is unicursal we can carry out the integration completely in exactly the same sense as in the case of rational functions. In particular, if the integral is *algebraical* then it can be found by means of elementary operations which are always practicable. And it has been shown, more generally, that we can always determine by means of such operations whether the integral of any given algebraical function is algebraical or not, and evaluate the integral when it is algebraical. And although the general problem of determining whether any given integral is an elementary function, and calculating it if it is one, has not been solved, the solution in the particular case in which the deficiency of the curve $f(x, y) = 0$ is unity is as complete as it is reasonable to expect any possible solution to be.

3. The theory of the integration of transcendental functions (VI.) is naturally much less complete, and the number of classes of such functions for which general methods of integration exist is very small. These few classes are, however, of extreme importance in applications (VI.; 2, 3).

There is a general theorem concerning the form of an integral of a transcendental function, when it is itself an elementary function, which is quite analogous to those already stated for rational and algebraical functions. The general statement of this theorem will be found in VI., § 5; it shows, for instance, that the integral of a rational function of x, e^x and $\log x$ is either a rational function of those functions or the sum of such a rational function and of a finite number of constant multiples of logarithms of similar functions. From this general theorem may be deduced a number of more precise results concerning integrals of more special forms, such as

$$\int y e^x \, dx, \quad \int y \log x \, dx,$$

where y is an algebraical function of x (VI.; 4, 6).

IV. Rational functions

1. It is proved in treatises on algebra* that any polynomial

$$Q(x) = b_0 x^n + b_1 x^{n-1} + \dots + b_n$$

can be expressed in the form

$$b_0 (x - a_1)^{n_1} (x - a_2)^{n_2} \dots (x - a_r)^{n_r},$$

where $n_1, n_2, \dots$ are positive integers whose sum is n, and $a_1, a_2, \dots$ are constants; and that any rational function $R(x)$, whose denominator is $Q(x)$, may be expressed in the form

$$A_0 x^p + A_1 x^{p-1} + \dots + A_p + \sum_{s=1}^{r} \left\{ \frac{\beta_{s,1}}{x - a_s} + \frac{\beta_{s,2}}{(x - a_s)^2} + \dots + \frac{\beta_{s,n_s}}{(x - a_s)^{n_s}} \right\},$$

where $A_0, A_1, \dots, \beta_{s,1}, \dots$ are also constants. It follows that

$$\int R(x)\, dx = A_0 \frac{x^{p+1}}{p+1} + A_1 \frac{x^p}{p} + \dots + A_p x + C$$
$$+ \sum_{s=1}^{r} \left\{ \beta_{s,1} \log(x - a_s) - \frac{\beta_{s,2}}{x - a_s} - \dots - \frac{\beta_{s,n_s}}{(n_s - 1)(x - a_s)^{n_s - 1}} \right\}.$$

From this we conclude that *the integral of any rational function is an elementary function which is rational save for the possible presence of logarithms of rational functions.* In particular the integral will be *rational* if each of the numbers $\beta_{s,1}$ is zero: this condition is evidently necessary and sufficient. A necessary but not sufficient condition is that $Q(x)$ should contain no simple factors.

The integral of the general rational function may be expressed in a very simple and elegant form by means of symbols of differentiation. We may suppose for simplicity that the degree of $P(x)$ is less than that of $Q(x)$; this can of course always be ensured by subtracting a polynomial from $R(x)$. Then

$$R(x) = \frac{P(x)}{Q(x)}$$
$$= \frac{1}{(n_1 - 1)!\,(n_2 - 1)! \dots (n_r - 1)!} \frac{\partial^{n-r}}{\partial a_1^{n_1 - 1} \partial a_2^{n_2 - 1} \dots \partial a_r^{n_r - 1}} \frac{P(x)}{Q_0(x)},$$

where $$Q_0(x) = b_0 (x - a_1)(x - a_2) \dots (x - a_r).$$

Now $$\frac{P(x)}{Q_0(x)} = \varpi_0(x) + \sum_{s=1}^{r} \frac{P(a_s)}{(x - a_s)\, Q_0'(a_s)},$$

* See, *e.g.*, Weber's *Traité d'algèbre supérieure* (French translation by J. Griess, Paris, 1898), vol. 1, pp. 61–64, 143–149, 350–353; or Chrystal's *Algebra*, vol. 1, pp. 151–162.

where $\varpi_0(x)$ is a polynomial; and so

$$\int R(x)\,dx$$
$$=\frac{1}{(n_1-1)!\dots(n_r-1)!}\frac{\partial^{n-r}}{\partial a_1^{n_1-1}\dots\partial a_r^{n_r-1}}\left[\Pi_0(x)+\sum_{s=1}^{r}\frac{P(a_s)}{Q_0'(a_s)}\log(x-a_s)\right],$$

where
$$\Pi_0(x)=\int\varpi_0(x)\,dx.$$

But
$$\Pi(x)=\frac{\partial^{n-r}\Pi_0(x)}{\partial a_1^{n_1-1}\partial a_2^{n_2-1}\dots\partial a_r^{n_r-1}}$$

is also a polynomial, and the integral contains no polynomial term, since the degree of $P(x)$ is less than that of $Q(x)$. Thus $\Pi(x)$ must vanish identically, so that

$$\int R(x)\,dx$$
$$=\frac{1}{(n_1-1)!\dots(n_r-1)!}\frac{\partial^{n-r}}{\partial a_1^{n_1-1}\dots\partial a_r^{n_r-1}}\left[\sum_{s=1}^{r}\frac{P(a_s)}{Q_0'(a_s)}\log(x-a_s)\right].$$

For example

$$\int\frac{dx}{\{(x-a)(x-b)\}^2}=\frac{\partial^2}{\partial a\,\partial b}\left\{\frac{1}{a-b}\log\left(\frac{x-a}{x-b}\right)\right\}.$$

That $\Pi_0(x)$ is annihilated by the partial differentiations performed on it may be verified directly as follows. We obtain $\Pi_0(x)$ by picking out from the expansion

$$\frac{P(x)}{x^r}\left(1+\frac{a_1}{x}+\frac{a_1^2}{x^2}+\dots\right)\left(1+\frac{a_2}{x}+\frac{a_2^2}{x^2}+\dots\right)\dots\dots$$

the terms which involve positive powers of x. Any such term is of the form

$$Ax^{\nu-r-s_1-s_2-\dots}a_1^{s_1}a_2^{s_2}\dots,$$

where
$$s_1+s_2+\dots\leqslant\nu-r\leqslant m-r,$$

m being the degree of P. It follows that

$$s_1+s_2+\dots<n-r=(m_1-1)+(m_2-1)+\dots;$$

so that at least one of $s_1, s_2, \dots$ must be less than the corresponding one of $m_1-1, m_2-1, \dots$.

It has been assumed above that if

$$F(x, a)=\int f(x, a)\,dx,$$

then
$$\frac{\partial F}{\partial a}=\int\frac{\partial f}{\partial a}\,dx.$$

The first equation means that $f=\frac{\partial F}{\partial x}$ and the second that $\frac{\partial f}{\partial a}=\frac{\partial^2 F}{\partial x \partial a}$. As it follows from the first that $\frac{\partial f}{\partial a}=\frac{\partial^2 F}{\partial a \partial x}$, what has really been assumed is that

$$\frac{\partial^2 F}{\partial a \partial x}=\frac{\partial^2 F}{\partial x \partial a}.$$

It is known that this equation is always true for $x=x_0$, $a=a_0$ if a circle can be drawn in the plane of (x, a) whose centre is (x_0, a_0) and within which the differential coefficients are continuous.

2. It appears from § 1 that the integral of a rational function is in general composed of two parts, one of which is a rational function and the other a function of the form

$$\Sigma A \log(x-\alpha) \quad \text{..........................(1).}$$

We may call these two functions the *rational part* and the *transcendental part* of the integral. It is evidently of great importance to show that the 'transcendental part' of the integral is really transcendental and cannot be expressed, wholly or in part, as a rational or algebraical function.

We are not yet in a position to prove this completely*; but we can take the first step in this direction by showing that *no sum of the form* (1) *can be rational, unless every A is zero.*

Suppose, if possible, that

$$\Sigma A \log(x-\alpha)=\frac{P(x)}{Q(x)} \quad \text{.....................(2),}$$

where P and Q are polynomials without common factor. Then

$$\Sigma \frac{A}{x-\alpha}=\frac{P'Q-PQ'}{Q^2} \quad \text{.........................(3).}$$

Suppose now that $(x-p)^r$ is a factor of Q. Then $P'Q-PQ'$ is divisible by $(x-p)^{r-1}$ and by no higher power of $x-p$. Thus the right-hand side of (3), when expressed in its lowest terms, has a factor $(x-p)^{r+1}$ in its denominator. On the other hand the left-hand side, when expressed as a rational fraction in its lowest terms, has no repeated factor in its denominator. Hence $r=0$, and so Q is a constant. We may therefore replace (2) by

$$\Sigma A \log(x-\alpha)=P(x),$$

and (3) by

$$\Sigma \frac{A}{x-\alpha}=P'(x).$$

Multiplying by $x-\alpha$, and making x tend to α, we see that $A=0$.

* The proof will be completed in v., 16.

3. The method of § 1 gives a complete solution of the problem if the roots of $Q(x)=0$ can be determined; and in practice this is usually the case. But this case, though it is the one which occurs most frequently in practice, is from a theoretical point of view an exceedingly special case. The roots of $Q(x)=0$ are not in general explicit algebraical functions of the coefficients, and cannot as a rule be determined in any explicit form. The method of partial fractions is therefore subject to serious limitations. For example, we cannot determine, by the method of decomposition into partial fractions, such an integral as

$$\int \frac{4x^9+21x^6+2x^3-3x^2-3}{(x^7-x+1)^2}\,dx,$$

or even determine whether the integral is rational or not, although it is in reality a very simple function. A high degree of importance therefore attaches to the further problem of determining the integral of a given rational function so far as possible in an absolutely explicit form and by means of operations which are always practicable.

It is easy to see that a complete solution of this problem cannot be looked for.

Suppose for example that $P(x)$ reduces to unity, and that $Q(x)=0$ is an equation of the fifth degree, whose roots $a_1, a_2, \dots a_5$ are all distinct and not capable of explicit algebraical expression.

Then
$$\int R(x)\,dx = \sum_1^5 \frac{\log(x-a_s)}{Q'(a_s)}$$
$$= \log \prod_1^5 \{(x-a_s)^{1/Q'(a_s)}\},$$

and it is only if at least two of the numbers $Q'(a_s)$ are commensurable that any two or more of the factors $(x-a_s)^{1/Q'(a_s)}$ can be associated so as to give a single term of the type $A \log S(x)$, where $S(x)$ is rational. In general this will not be the case, and so it will not be possible to express the integral in any finite form which does not explicitly involve the roots. A more precise result in this connection will be proved later (§ 6).

4. The first and most important part of the problem has been solved by Hermite, who has shown that the *rational part* of the integral can always be determined without a knowledge of the roots of $Q(x)$, and indeed without the performance of any operations other than those of elementary algebra*.

* The following account of Hermite's method is taken in substance from Goursat's *Cours d'analyse mathématique* (first edition), t. 1, pp. 238–241.

Hermite's method depends upon a fundamental theorem in elementary algebra* which is also of great importance in the ordinary theory of partial fractions, viz.:

'*If* X_1 *and* X_2 *are two polynomials in* x *which have no common factor, and* X_3 *any third polynomial, then we can determine two polynomials* A_1, A_2, *such that*

$$A_1X_1 + A_2X_2 = X_3.\text{'}$$

Suppose that $$Q(x) = Q_1Q_2^2Q_3^3\dots Q_t^t,$$

$Q_1, \dots$ denoting polynomials which have only simple roots and of which no two have any common factor. We can always determine $Q_1, \dots$ by elementary methods, as is shown in the elements of the theory of equations†.

We can determine B and A_1 so that

$$BQ_1 + A_1Q_2^2Q_3^3\dots Q_t^t = P,$$

and therefore so that

$$R(x) = \frac{P}{Q} = \frac{A_1}{Q_1} + \frac{B}{Q_2^2Q_3^3\dots Q_t^t}.$$

By a repetition of this process we can express $R(x)$ in the form

$$\frac{A_1}{Q_1} + \frac{A_2}{Q_2^2} + \dots + \frac{A_t}{Q_t^t},$$

and the problem of the integration of $R(x)$ is reduced to that of the integration of a function

$$\frac{A}{Q^\nu},$$

where Q is a polynomial whose roots are all distinct. Since this is so, Q and its derived function Q' have no common factor: we can therefore determine C and D so that

$$CQ + DQ' = A.$$

Hence

$$\begin{aligned}\int \frac{A}{Q^\nu}\,dx &= \int \frac{CQ + DQ'}{Q^\nu}\,dx \\ &= \int \frac{C}{Q^{\nu-1}}\,dx - \frac{1}{\nu-1}\int D\,\frac{d}{dx}\left(\frac{1}{Q^{\nu-1}}\right)dx \\ &= -\frac{D}{(\nu-1)\,Q^{\nu-1}} + \int \frac{E}{Q^{\nu-1}}\,dx,\end{aligned}$$

where $$E = C + \frac{D'}{\nu-1}.$$

* See Chrystal's *Algebra*, vol. 1, pp. 119 *et seq.*

† See, for example, Hardy, *A course of pure mathematics* (2nd edition), p. 208.

Proceeding in this way, and reducing by unity at each step the power of $1/Q$ which figures under the sign of integration, we ultimately arrive at an equation

$$\int \frac{A}{Q^\nu}\,dx = R_\nu(x) + \int \frac{S}{Q}\,dx,$$

where R_ν is a rational function and S a polynomial.

The integral on the right-hand side has no rational part, since all the roots of Q are simple (§ 2). Thus the rational part of $\int R(x)\,dx$ is

$$R_2(x) + R_3(x) + \dots + R_t(x),$$

and it has been determined without the need of any calculations other than those involved in the addition, multiplication and division of polynomials*.

5. (i) Let us consider, for example, the integral

$$\int \frac{4x^9+21x^6+2x^3-3x^2-3}{(x^7-x+1)^2}\,dx,$$

mentioned above (§ 3). We require polynomials A_1, A_2 such that

$$A_1X_1+A_2X_2=X_3 \quad \text{..............................(1)},$$

where

$$X_1=x^7-x+1, \quad X_2=7x^6-1, \quad X_3=4x^9+21x^6+2x^3-3x^2-3.$$

In general, if the degrees of X_1 and X_2 are m_1 and m_2, and that of X_3 does not exceed m_1+m_2-1, we can suppose that the degrees of A_1 and A_2 do not exceed m_2-1 and m_1-1 respectively. For we know that polynomials B_1 and B_2 exist such that

$$B_1X_1+B_2X_2=X_3.$$

If B_1 is of degree not exceeding m_2-1, we take $A_1=B_1$, and if it is of higher degree we write

$$B_1=L_1X_2+A_1,$$

where A_1 is of degree not exceeding m_2-1. Similarly we write

$$B_2=L_2X_1+A_2.$$

We have then

$$(L_1+L_2)\,X_1X_2+A_1X_1+A_2X_2=X_3.$$

In this identity L_1 or L_2 or both may vanish identically, and in any case we see, by equating to zero the coefficients of the powers of x higher than the (m_1+m_2-1)th, that L_1+L_2 vanishes identically. Thus X_3 is expressed in the form required.

The actual determination of the coefficients in A_1 and A_2 is most easily performed by equating coefficients. We have then m_1+m_2 linear equations

* The operation of forming the derived function of a given polynomial can of course be effected by a combination of these operations.

in the same number of unknowns. These equations must be consistent, since we know that a solution exists*.

If X_3 is of degree higher than m_1+m_2-1, we must divide it by X_1X_2 and express the remainder in the form required.

In this case we may suppose A_1 of degree 5 and A_2 of degree 6, and we find that

$$A_1=-3x^2, \quad A_2=x^3+3.$$

Thus the rational part of the integral is

$$-\frac{x^3+3}{x^7-x+1},$$

and, since $-3x^2+(x^3+3)'=0$, there is no transcendental part.

(ii) The following problem is instructive: *to find the conditions that*

$$\int \frac{ax^2+2\beta x+\gamma}{(Ax^2+2Bx+C)^2}\,dx$$

may be rational, and to determine the integral when it is rational.

We shall suppose that $Ax^2+2Bx+C$ is not a perfect square, as if it were the integral would certainly be rational. We can determine p, q and r so that

$$p(Ax^2+2Bx+C)+2(qx+r)(Ax+B)=ax^2+2\beta x+\gamma,$$

and the integral becomes

$$p\int\frac{dx}{Ax^2+2Bx+C}-\int(qx+r)\frac{d}{dx}\left(\frac{1}{Ax^2+2Bx+C}\right)dx$$
$$=-\frac{qx+r}{Ax^2+2Bx+C}+(p+q)\int\frac{dx}{Ax^2+2Bx+C}.$$

The condition that the integral should be rational is therefore $p+q=0$.

Equating coefficients we find

$$A(p+2q)=a, \quad B(p+q)+Ar=\beta, \quad Cp+2Br=\gamma.$$

Hence we deduce

$$p=-\frac{a}{A}, \qquad q=\frac{a}{A}, \qquad r=\frac{\beta}{A},$$

and $A\gamma+Ca=2B\beta$. The condition required is therefore that the two quadratics $ax^2+2\beta x+\gamma$ and $Ax^2+2Bx+C$ should be harmonically related, and in this case

$$\int\frac{ax^2+2\beta x+\gamma}{(Ax^2+2Bx+C)^2}\,dx=-\frac{ax+\beta}{A(Ax^2+2Bx+C)}.$$

(iii) Another method of solution of this problem is as follows. If we write

$$Ax^2+2Bx+C=A(x-\lambda)(x-\mu),$$

and use the bilinear substitution

$$x=\frac{\lambda y+\mu}{y+1},$$

then the integral is reduced to one of the form

$$\int\frac{ay^2+2by+c}{y^2}\,dy,$$

* It is easy to show that the solution is also unique.

and is rational if and only if $b=0$. But this is the condition that the quadratic $ay^2+2by+c$, corresponding to $ax^2+2\beta x+\gamma$, should be harmonically related to the degenerate quadratic y, corresponding to $Ax^2+2Bx+C$. The result now follows from the fact that harmonic relations are not changed by bilinear transformation.

It is not difficult to show, by an adaptation of this method, that

$$\int \frac{(ax^2+2\beta x+\gamma)(a_1x^2+2\beta_1x+\gamma_1)\dots(a_nx^2+2\beta_nx+\gamma_n)}{(Ax^2+2Bx+C)^{n+2}}\,dx$$

is rational if all the quadratics are harmonically related to any one of those in the numerator. This condition is sufficient but not necessary.

(iv) As a further example of the use of the method (ii) the reader may show that *the necessary and sufficient condition that*

$$\int \frac{f(x)}{\{F(x)\}^2}\,dx,$$

where f and F are polynomials with no common factor, and F has no repeated factor, should be rational, is that $f'F'-fF''$ should be divisible by F.

6. It appears from the preceding paragraphs that we can always find the rational part of the integral, and can find the complete integral if we can find the roots of $Q(x)=0$. The question is naturally suggested as to the maximum of information which can be obtained about the logarithmic part of the integral in the general case in which the factors of the denominator cannot be determined explicitly. For there are polynomials which, although they cannot be completely resolved into such factors, can nevertheless be partially resolved. For example

$$x^{14}-2x^8-2x^7-x^4-2x^3+2x+1=(x^7+x^2-1)(x^7-x^2-2x-1),$$

$$x^{14}-2x^8-2x^7-2x^4-4x^3-x^2+2x+1$$
$$=\{x^7+x^2\sqrt{2}+x(\sqrt{2}-1)-1\}\{x^7-x^2\sqrt{2}-x(\sqrt{2}+1)-1\}.$$

The factors of the first polynomial have rational coefficients: in the language of the theory of equations, the polynomial is *reducible in the rational domain.* The second polynomial is reducible in the domain formed by the *adjunction* of the single irrational $\sqrt{2}$ to the rational domain*.

We may suppose that every possible decomposition of $Q(x)$ of this nature has been made, so that

$$Q=Q_1Q_2\dots Q_t.$$

* See Cajori, *An introduction to the modern theory of equations* (Macmillan, 1904); Mathews, *Algebraic equations* (*Cambridge tracts in mathematics*, no. 6), pp. 6–7.

Then we can resolve $R(x)$ into a sum of partial fractions of the type

$$\int \frac{P_\nu}{Q_\nu} dx,$$

and so we need only consider integrals of the type

$$\int \frac{P}{Q} dx,$$

where no further resolution of Q is possible or, in technical language, *Q is irreducible by the adjunction of any algebraical irrationality.*

Suppose that this integral can be evaluated in a form involving only constants which can be expressed explicitly in terms of the constants which occur in P/Q. It must be of the form

$$A_1 \log X_1 + \ldots + A_k \log X_k \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where the A's are constants and the X's polynomials. We can suppose that no X has any repeated factor ξ^m, where ξ is a polynomial. For such a factor could be determined rationally in terms of the coefficients of X, and the expression (1) could then be modified by taking out the factor ξ^m from X and inserting a new term $mA \log \xi$. And for similar reasons we can suppose that no two X's have any factor in common.

Now
$$\frac{P}{Q} = A_1 \frac{X_1'}{X_1} + A_2 \frac{X_2'}{X_2} + \ldots + A_k \frac{X_k'}{X_k},$$

or
$$PX_1X_2 \ldots X_k = Q \Sigma A_\nu X_1 \ldots X_{\nu-1} X_\nu' X_{\nu+1} \ldots X_k.$$

All the terms under the sign of summation are divisible by X_1 save the first, which is prime to X_1. Hence Q must be divisible by X_1: and similarly, of course, by $X_2, X_3, \ldots, X_k$. But, since P is prime to Q, $X_1X_2 \ldots X_k$ is divisible by Q. Thus Q must be a constant multiple of $X_1X_2 \ldots X_k$. But Q is *ex hypothesi* not resoluble into factors which contain only explicit algebraical irrationalities. Hence all the X's save one must reduce to constants, and so P must be a constant multiple of Q', and

$$\int \frac{P}{Q} dx = A \log Q,$$

where A is a constant. Unless this is the case the integral cannot be expressed in a form involving only constants expressed explicitly in terms of the constants which occur in P and Q.

Thus, for instance, the integral

$$\int \frac{dx}{x^5 + ax + b}$$

cannot, except in special cases*, be expressed in a form involving only constants expressed explicitly in terms of a and b; and the integral

$$\int \frac{5x^4+c}{x^5+ax+b}\,dx$$

can in general be so expressed if and only if $c=a$. We thus confirm an inference made before (§ 3) in a less accurate way.

Before quitting this part of our subject we may consider one further problem: *under what circumstances is*

$$\int R(x)\,dx = A \log R_1(x)$$

where A is a constant and R_1 rational? Since the integral has no rational part, it is clear that $Q(x)$ must have only simple factors, and that the degree of $P(x)$ must be less than that of $Q(x)$. We may therefore use the formula

$$\int R(x)\,dx = \log \prod_1^r \{(x-a_s)^{P(a_s)/Q'(a_s)}\}.$$

The necessary and sufficient condition is that all the numbers $P(a_s)/Q'(a_s)$ should be commensurable. If *e.g.*

$$R(x) = \frac{x-\gamma}{(x-a)(x-\beta)},$$

then $(a-\gamma)/(a-\beta)$ and $(\beta-\gamma)/(\beta-a)$ must be commensurable, *i.e.* $(a-\gamma)/(\beta-\gamma)$ must be a rational number. If the denominator is given we can find all the values of γ which are admissible: for $\gamma=(aq-\beta p)/(q-p)$, where p and q are integers.

7. Our discussion of the integration of rational functions is now complete. It has been throughout of a theoretical character. We have not attempted to consider what are the simplest and quickest methods for the actual calculation of the types of integral which occur most commonly in practice. This problem lies outside our present range: the reader may consult

O. Stolz, *Grundzüge der Differential-und-integralrechnung*, vol. 1, ch. 7:

J. Tannery, *Leçons d'algèbre et d'analyse*, vol. 2, ch. 18:

Ch.-J. de la Vallée-Poussin, *Cours d'analyse*, ed. 3, vol. 1, ch. 5:

T. J. I'A. Bromwich, *Elementary integrals* (Bowes and Bowes, 1911):

G. H. Hardy, *A course of pure mathematics*, ed. 2, ch. 6.

* The equation $x^5+ax+b=0$ is soluble by radicals in certain cases. See Mathews, *l.c.*, pp. 52 *et seq.*

V. Algebraical Functions

1. We shall now consider the integrals of algebraical functions, explicit or implicit. The theory of the integration of such functions is far more extensive and difficult than that of rational functions, and we can give here only a brief account of a few of the most important results and of the most obvious of their applications.

If $y_1, y_2, \ldots, y_n$ are algebraical functions of x, then any algebraical function z of $x, y_1, \ldots, y_n$ is an algebraical function of x. This is obvious if we confine ourselves to *explicit* algebraical functions. In the general case we have a number of equations of the type

$$P_{\nu,0}(x)\, y_\nu^{m_\nu} + P_{\nu,1}(x)\, y_\nu^{m_\nu - 1} + \ldots + P_{\nu,m_\nu}(x) = 0 \quad (\nu = 1, 2, \ldots, n),$$

and $$P_0(x, y_1, \ldots, y_n)\, z^m + \ldots + P_m(x, y_1, \ldots, y_n) = 0,$$

where the P's represent polynomials in their arguments. The elimination of $y_1, y_2, \ldots, y_n$ between these equations gives an equation in z whose coefficients are polynomials in x only.

The importance of this from our present point of view lies in the fact that we may consider the standard algebraical integral under any of the forms

$$\int y\, dx,$$

where $f(x, y) = 0$;

$$\int R(x, y)\, dx,$$

where $f(x, y) = 0$ and R is rational; or

$$\int R(x, y_1, \ldots, y_n)\, dx,$$

where $f_1(x, y) = 0, \ldots, f_n(x, y_n) = 0$. It is, for example, much more convenient to treat such an irrational as

$$\frac{x - \surd(x+1) - \surd(x-1)}{1 + \surd(x+1) + \surd(x-1)}$$

as a rational function of x, y_1, y_2, where $y_1 = \surd(x+1)$, $y_2 = \surd(x-1)$, $y_1^2 = x + 1$, $y_2^2 = x - 1$, than as a rational function of x and y, where

$$y = \surd(x+1) + \surd(x-1),$$
$$y^4 - 4xy^2 + 4 = 0.$$

To treat it as a simple irrational y, so that our fundamental equation is

$$(x - y)^4 - 4x(x - y)^2(1 + y)^2 + 4(1 + y)^4 = 0$$

is evidently the least convenient course of all.

Before we proceed to consider the general form of the integral of an algebraical function we shall consider one most important case in which the integral can be at once reduced to that of a rational function, and is therefore always an elementary function itself.

2. The class of integrals alluded to immediately above is that covered by the following theorem.

If there is a variable t connected with x and y (or $y_1, y_2, \dots, y_n$) by rational relations

$$x = R_1(t), \quad y = R_2(t)$$

(or $y_1 = R_2^{(1)}(t)$, $y_2 = R_2^{(2)}(t)$, ...), then the integral

$$\int R(x, y)\, dx$$

(or $\int R(x, y_1, \dots, y_n)\, dx$) is an elementary function.

The truth of this proposition follows immediately from the equations

$$R(x, y) = R\{R_1(t), R_2(t)\} = S(t),$$

$$\frac{dx}{dt} = R_1'(t) = T(t),$$

$$\int R(x, y)\, dx = \int S(t)\ T(t)\, dt = \int U(t)\, dt,$$

where all the capital letters denote rational functions.

The most important case of this theorem is that in which x and y are connected by the general quadratic relation

$$(a, b, c, f, g, h \between x, y, 1)^2 = 0.$$

The integral can then be made rational in an infinite number of ways. For suppose that (ξ, η) is any point on the conic, and that

$$(y - \eta) = t(x - \xi)$$

is any line through the point. If we eliminate y between these equations, we obtain an equation of the second degree in x, say

$$T_0 x^2 + 2T_1 x + T_2 = 0,$$

where T_0, T_1, T_2 are polynomials in t. But one root of this equation must be ξ, which is independent of t; and when we divide by $x - \xi$ we obtain an equation of the *first* degree for the abscissa of the variable point of intersection, in which the coefficients are again polynomials in t. Hence this abscissa is a rational function of t; the ordinate of the point is also a rational function of t, and as t varies this point coincides with every point of the conic in turn. In fact the equation of the conic may be written in the form

$$au^2 + 2huv + bv^2 + 2(a\xi + h\eta + g)u + 2(h\xi + b\eta + f)v = 0,$$

where $u = x - \xi$, $v = y - \eta$, and the other point of intersection of the line $v = tu$ and the conic is given by

$$x = \xi - \frac{2\{a\xi + h\eta + g + t(h\xi + b\eta + f)\}}{a + 2ht + bt^2},$$

$$y = \eta - \frac{2t\{a\xi + h\eta + g + t(h\xi + b\eta + f)\}}{a + 2ht + bt^2}.$$

An alternative method is to write

$$ax^2 + 2hxy + by^2 = b(y - \mu x)(y - \mu' x),$$

so that $y - \mu x = 0$ and $y - \mu' x = 0$ are parallel to the asymptotes of the conic, and to put

$$y - \mu x = t.$$

Then $$y - \mu' x = -\frac{2gx + 2fy + c}{bt};$$

and from these two equations we can calculate x and y as rational functions of t. The principle of this method is of course the same as that of the former method: (ξ, η) is now at infinity, and the pencil of lines through (ξ, η) is replaced by a pencil parallel to an asymptote.

The most important case is that in which $b = -1$, $f = h = 0$, so that

$$y^2 = ax^2 + 2gx + c.$$

The integral is then made rational by the substitution

$$x = \xi - \frac{2(a\xi + g - t\eta)}{a - t^2}, \quad y = \eta - \frac{2t(a\xi + g - t\eta)}{a - t^2},$$

where ξ, η are any numbers such that

$$\eta^2 = a\xi^2 + 2g\xi + c.$$

We may for instance suppose that $\xi = 0$, $\eta = \sqrt{c}$; or that $\eta = 0$, while ξ is a root of the equation $a\xi^2 + 2g\xi + c = 0$. Or again the integral is made rational by putting $y - x\sqrt{a} = t$, when

$$x = -\frac{t^2 - c}{2(t\sqrt{a} - g)}, \quad y = \frac{(t^2 + c)\sqrt{a} - 2gt}{2(t\sqrt{a} - g)}.$$

3. We shall now consider in more detail the problem of the calculation of

$$\int R(x, y)\, dx,$$

where $$y = \sqrt{X} = \sqrt{(ax^2 + 2bx + c)}\,^*.$$

* We now write b for g for the sake of symmetry in notation.

The most interesting case is that in which a, b, c and the constants which occur in R are real, and we shall confine our attention to this case.

Let
$$R(x, y) = \frac{P(x, y)}{Q(x, y)},$$
where P and Q are polynomials. Then, by means of the equation
$$y^2 = ax^2 + 2bx + c,$$
$R(x, y)$ may be reduced to the form
$$\frac{A + B\sqrt{X}}{C + D\sqrt{X}} = \frac{(A + B\sqrt{X})(C - D\sqrt{X})}{C^2 - D^2 X},$$
where A, B, C, D are polynomials in x; and so to the form $M + N\sqrt{X}$, where M and N are rational, or (what is the same thing) the form
$$P + \frac{Q}{\sqrt{X}},$$
where P and Q are rational. The rational part may be integrated by the methods of section IV., and the integral
$$\int \frac{Q}{\sqrt{X}}\, dx$$
may be reduced to the sum of a number of integrals of the forms
$$\int \frac{x^r}{\sqrt{X}}\, dx, \quad \int \frac{dx}{(x-p)^r \sqrt{X}}, \quad \int \frac{\xi x + \eta}{(ax^2 + 2\beta x + \gamma)^r \sqrt{X}}\, dx \quad \text{......(1)},$$
where p, ξ, η, a, β, γ are real constants and r a positive integer. The result is generally required in an explicitly real form: and, as further progress depends on transformations involving p (or a, β, γ), it is generally not advisable to break up a quadratic factor $ax^2 + 2\beta x + \gamma$ into its constituent linear factors when these factors are complex.

All of the integrals (1) may be reduced, by means of elementary formulae of reduction*, to dependence upon three fundamental integrals, viz.
$$\int \frac{dx}{\sqrt{X}}, \quad \int \frac{dx}{(x-p)\sqrt{X}}, \quad \int \frac{\xi x + \eta}{(ax^2 + 2\beta x + \gamma)\sqrt{X}}\, dx \quad \text{.........(2)}.$$

4. The first of these integrals may be reduced, by a substitution of the type $x = t + k$, to one or other of the three standard forms
$$\int \frac{dt}{\sqrt{(m^2 - t^2)}}, \quad \int \frac{dt}{\sqrt{(t^2 + m^2)}}, \quad \int \frac{dt}{\sqrt{(t^2 - m^2)}},$$
where $m > 0$. These integrals may be rationalised by the substitutions
$$t = \frac{2mu}{1 + u^2}, \quad t = \frac{2mu}{1 - u^2}, \quad t = \frac{m(1 + u^2)}{2u};$$
but it is simpler to use the transcendental substitutions
$$t = m \sin\phi, \quad t = m \sinh\phi, \quad t = m \cosh\phi.$$

* See, for example, Bromwich, *l.c.*, pp. 16 *et seq.*

These last substitutions are generally the most convenient for the reduction of an integral which contains one or other of the irrationalities

$$\sqrt{(m^2-t^2)}, \qquad \sqrt{(t^2+m^2)}, \qquad \sqrt{(t^2-m^2)},$$

though the alternative substitutions

$$t=m\tanh\phi, \qquad t=m\tan\phi, \qquad t=m\sec\phi$$

are often useful.

It has been pointed out by Dr Bromwich that the forms usually given in text-books for these three standard integrals, viz.

$$\operatorname{arc\,sin}\frac{t}{m}, \qquad \operatorname{arg\,sinh}\frac{t}{m}, \qquad \operatorname{arg\,cosh}\frac{t}{m},$$

are not quite accurate. It is obvious, for example, that the first two of these functions are odd functions of m, while the corresponding integrals are even functions. The correct formulae are

$$\operatorname{arc\,sin}\frac{t}{|m|}, \qquad \operatorname{arg\,sinh}\frac{t}{|m|}=\log\frac{t+\sqrt{(t^2+m^2)}}{|m|}$$

and

$$\pm\operatorname{arg\,cosh}\frac{|t|}{|m|}=\log\left|\frac{t+\sqrt{(t^2-m^2)}}{m}\right|,$$

where the ambiguous sign is the same as that of t. It is in some ways more convenient to use the equivalent forms

$$\operatorname{arc\,tan}\frac{t}{\sqrt{(m^2-t^2)}}, \qquad \operatorname{arg\,tanh}\frac{t}{\sqrt{(t^2+m^2)}}, \qquad \operatorname{arg\,tanh}\frac{t}{\sqrt{(t^2-m^2)}}.$$

5. The integral

$$\int\frac{dx}{(x-p)\sqrt{X}}$$

may be evaluated in a variety of ways.

If p is a root of the equation $X=0$, then X may be written in the form $a(x-p)(x-q)$, and the value of the integral is given by one or other of the formulae

$$\int\frac{dx}{(x-p)\sqrt{\{(x-p)(x-q)\}}}=\frac{2}{q-p}\sqrt{\left(\frac{x-q}{x-p}\right)},$$

$$\int\frac{dx}{(x-p)^{5/2}}=-\frac{2}{3(x-p)^{3/2}}.$$

We may therefore suppose that p is not a root of $X=0$.

(i) We may follow the general method described above, taking

$$\xi=p, \qquad \eta=\sqrt{(ap^2+2bp+c)}\text{*}.$$

Eliminating y from the equations

$$y^2=ax^2+2bx+c, \qquad y-\eta=t(x-\xi),$$

and dividing by $x-\xi$, we obtain

$$t^2(x-\xi)+2\eta t-a(x+\xi)-2b=0,$$

and so

$$-\frac{2dt}{t^2-a}=\frac{dx}{t(x-\xi)+\eta}=\frac{dx}{y}.$$

* Cf. Jordan, *Cours d'analyse*, ed. 2, vol. 2, p. 21.

Hence $$\int \frac{dx}{(x-\xi)\,y} = -2\int \frac{dt}{(x-\xi)(t^2-a)}.$$

But $$(t^2-a)(x-\xi) = 2a\xi + 2b - 2\eta t;$$

and so

$$\int \frac{dx}{(x-p)\,y} = -\int \frac{dt}{a\xi+b-\eta t} = \frac{1}{\eta}\log(a\xi+b-\eta t)$$

$$= \frac{1}{\sqrt{(ap^2+2bp+c)}}\log\{t\sqrt{(ap^2+2bp+c)} - ap - b\}.$$

If $ap^2+2bp+c<0$ the transformation is imaginary.

Suppose, *e.g.*, (*a*) $y=\sqrt{(x+1)}$, $p=0$, or (*b*) $y=\sqrt{(x-1)}$, $p=0$. We find

(*a*) $$\int \frac{dx}{x\sqrt{(x+1)}} = \log(t-\tfrac{1}{2}),$$

where $$t^2x + 2t - 1 = 0,$$

or $$t = \frac{-1+\sqrt{(x+1)}}{x};$$

and

(*b*) $$\int \frac{dx}{x\sqrt{(x-1)}} = -i\log(it-\tfrac{1}{2}),$$

where $$t^2x + 2it - 1 = 0.$$

Neither of these results is expressed in the simplest form, the second in particular being very inconvenient.

(ii) The most straightforward method of procedure is to use the substitution

$$x-p=\frac{1}{t}.$$

We then obtain

$$\int \frac{dx}{(x-p)\,y} = \int \frac{dt}{\sqrt{(a_1t^2+2b_1t+c_1)}},$$

where a_1, b_1, c_1 are certain simple functions of a, b, c, and p. The further reduction of this integral has been discussed already.

(iii) A third method of integration is that adopted by Sir G. Greenhill*, who uses the transformation

$$t=\frac{\sqrt{(ax^2+2bx+c)}}{x-p}.$$

It will be found that

$$\int \frac{dx}{(x-p)\sqrt{X}} = \int \frac{dt}{\sqrt{\{(ap^2+2bp+c)\,t^2+b^2-ac\}}},$$

which is of one of the three standard forms mentioned in § 4.

* A. G. Greenhill, *A chapter in the integral calculus* (Francis Hodgson, 1888), p. 12 : *Differential and integral calculus*, p. 399.

6. It remains to consider the integral

$$\int \frac{\xi x+\eta}{(ax^2+2\beta x+\gamma)\sqrt{X}}\,dx=\int \frac{\xi x+\eta}{X_1\sqrt{X}}\,dx,$$

where $ax^2+2\beta x+\gamma$ or X_1 is a quadratic with complex linear factors. Here again there is a choice of methods at our disposal.

We may suppose that X_1 is not a constant multiple of X. If it is, then the value of the integral is given by the formula

$$\int \frac{\xi x+\eta}{(ax^2+2bx+c)^{3/2}}\,dx=\frac{\eta(ax+b)-\xi(bx+c)}{\sqrt{\{(ac-b^2)(ax^2+2bx+c)\}}}\;^{*}.$$

(i) The standard method is to use the substitution

$$x=\frac{\mu t+\nu}{t+1} \quad\text{.....................................(1)},$$

where μ and ν are so chosen that

$$a\mu\nu+b(\mu+\nu)+c=0,\qquad a\mu\nu+\beta(\mu+\nu)+\gamma=0 \quad\text{............(2)}.$$

The values of μ and ν which satisfy these conditions are the roots of the quadratic

$$(a\beta-ba)\mu^2-(ca-a\gamma)\mu+(b\gamma-c\beta)=0.$$

The roots will be real and distinct if

$$(ca-a\gamma)^2>4(a\beta-ba)(b\gamma-c\beta),$$

or if

$$(a\gamma+ca-2b\beta)^2>4(ac-b^2)(a\gamma-\beta^2) \quad\text{..................(3)}.$$

Now $a\gamma-\beta^2>0$, so that (3) is certainly satisfied if $ac-b^2<0$. But if $ac-b^2$ and $a\gamma-\beta^2$ are both positive then $a\gamma$ and ca have the same sign, and

$$\begin{aligned}(a\gamma+ca-2b\beta)^2 &\geqslant (|a\gamma+ca|-2|b\beta|)^2>4\{\sqrt{(aca\gamma)}-|b\beta|\}^2\\ &=4[(ac-b^2)(a\gamma-\beta^2)+\{|b|\sqrt{(a\gamma)}-|\beta|\sqrt{(ac)}\}^2]\\ &\geqslant 4(ac-b^2)(a\gamma-\beta^2).\end{aligned}$$

Thus the values of μ and ν are in any case real and distinct.

It will be found, on carrying out the substitution (1), that

$$\int \frac{\xi x+\eta}{X_1\sqrt{X}}\,dx=H\int \frac{t\,dt}{(\mathbf{A}t^2+\mathbf{B})\sqrt{(At^2+B)}}+K\int \frac{dt}{(\mathbf{A}t^2+\mathbf{B})\sqrt{(At^2+B)}},$$

where $\mathbf{A}$, $\mathbf{B}$, A, B, H, and K are constants. Of these two integrals, the first is rationalised by the substitution

$$\frac{t}{\sqrt{(At^2+B)}}=u,$$

and the second by the substitution

$$\frac{1}{\sqrt{(At^2+B)}}=v.\dagger$$

It should be observed that this method fails in the special case in which

* Bromwich, *l.c.*, p. 16.

† The method sketched here is that followed by Stolz (see the references given on p. 21). Dr Bromwich's method is different in detail but the same in principle.

$a\beta - ba = 0$. In this case, however, the substitution $ax+b=t$ reduces the integral to one of the form

$$\int \frac{Ht+K}{(\mathbf{A}t^2+\mathbf{B})\sqrt{(At^2+B)}}\,dt,$$

and the reduction may then be completed as before.

(ii) An alternative method is to use Sir G. Greenhill's substitution

$$t=\sqrt{\left(\frac{X}{\alpha x^2+2\beta x+\gamma}\right)}=\sqrt{\left(\frac{X}{X_1}\right)}.$$

If $$J=(a\beta - b\alpha)\,x^2-(c\alpha - a\gamma)\,x+(b\gamma - c\beta),$$

then $$\frac{1}{t}\frac{dt}{dx}=\frac{J}{XX_1} \quad \text{..............................(1).}$$

The maximum and minimum values of t are given by $J=0$.

Again $$t^2-\lambda=\frac{(a-\lambda\alpha)\,x^2+2\,(b-\lambda\beta)\,x+(c-\lambda\gamma)}{X_1};$$

and the numerator will be a perfect square if

$$K=(\alpha\gamma-\beta^2)\,\lambda^2-(a\gamma+c\alpha-2b\beta)\,\lambda+(ac-b^2)=0.$$

It will be found by a little calculation that the discriminant of this quadratic and that of J differ from one another and from

$$(\phi-\phi_1)\,(\phi-\phi_1')\,(\phi'-\phi_1)\,(\phi'-\phi_1'),$$

where ϕ, ϕ' are the roots of $X=0$ and ϕ_1, ϕ_1' those of $X_1=0$, only by a constant factor which is always negative. Since ϕ_1 and ϕ_1' are conjugate complex numbers, this product is positive, and so $J=0$ and $K=0$ have real roots*. We denote the roots of the latter by

$$\lambda_1,\ \lambda_2 \qquad (\lambda_1>\lambda_2).$$

Then $$\lambda_1-t^2=\frac{\{x\sqrt{(\lambda_1\alpha-a)}+\sqrt{(\lambda_1\gamma-c)}\}^2}{X_1}=\frac{(mx+n)^2}{X_1} \quad \text{..............(2),}$$

$$t^2-\lambda_2=\frac{\{x\sqrt{(a-\lambda_2\alpha)}+\sqrt{(c-\lambda_2\gamma)}\}^2}{X_1}=\frac{(m'x+n')^2}{X_1} \quad \text{..............(2'),}$$

say. Further, since $t^2-\lambda$ can vanish for two equal values of x only if λ is equal to λ_1 or λ_2, *i.e.* when t is a maximum or a minimum, J can differ from

$$(mx+n)\,(m'x+n')$$

only by a constant factor; and by comparing coefficients and using the identity

$$(\lambda_1\alpha-a)\,(a-\lambda_2\alpha)=\frac{(a\beta-b\alpha)^2}{\alpha\gamma-\beta^2},$$

we find that $$J=\sqrt{(\alpha\gamma-\beta^2)}\,(mx+n)\,(m'x+n') \quad \text{....................(3).}$$

Finally, we can write $\xi x+\eta$ in the form

$$A\,(mx+n)+B\,(m'x+n').$$

* That the roots of $J=0$ are real has been proved already (p. 28) in a different manner.

Using equations (1), (2), (2′), and (3), we find that

$$\int \frac{\xi x+\eta}{X_1\sqrt{X}}\,dx=\int \frac{A\,(mx+n)+B\,(m'x+n')}{J}\sqrt{X_1}\,dt$$
$$=\frac{A}{\sqrt{(\alpha\gamma-\beta^2)}}\int\frac{dt}{\sqrt{(\lambda_1-t^2)}}+\frac{B}{\sqrt{(\alpha\gamma-\beta^2)}}\int\frac{dt}{\sqrt{(t^2-\lambda_2)}},$$

and the integral is reduced to a sum of two standard forms.

This method is very elegant, and has the advantage that the whole work of transformation is performed in one step. On the other hand it is somewhat artificial, and it is open to the logical objection that it introduces the root $\sqrt{X_1}$, which, in virtue of Laplace's principle (III., 2), cannot really be involved in the final result*.

7. We may now proceed to consider the general case to which the theorem of IV., § 2 applies. It will be convenient to recall two well-known definitions in the theory of algebraical plane curves. A curve of degree n can have at most $\frac{1}{2}(n-1)(n-2)$ double points†. If the actual number of double points is ν, then the number

$$p=\tfrac{1}{2}(n-1)(n-2)-\nu$$

is called the *deficiency* ‡ of the curve.

If the coordinates x, y of the points on a curve can be expressed *rationally* in terms of a parameter t by means of equations

$$x=R_1(t),\quad y=R_2(t),$$

then we shall say that the curve is *unicursal.* In this case we have seen that we can always evaluate

$$\int R\,(x, y)\,dx$$

in terms of elementary functions.

The fundamental theorem in this part of our subject is

'*A curve whose deficiency is zero is unicursal, and vice versa*'.

Suppose first that the curve possesses the maximum number of double points§. Since

$$\tfrac{1}{2}(n-1)(n-2)+n-3=\tfrac{1}{2}(n-2)(n+1)-1,$$

* The superfluous root may be eliminated from the result by a trivial transformation, just as $\sqrt{(1+x^2)}$ may be eliminated from

$$\text{arc sin}\,\frac{x}{\sqrt{(1+x^2)}}$$

by writing this function in the form arc tan x.

† Salmon, *Higher plane curves*, p. 29.

‡ Salmon, *ibid.*, p. 29. French *genre*, German *Geschlecht*.

§ We suppose in what follows that the singularities of the curve are all ordinary nodes. The necessary modifications when this is not the case are not difficult to and $\frac{1}{2}(n-2)(n+1)$ points are just sufficient to determine a curve of degree $n-2$*, we can draw, through the $\frac{1}{2}(n-1)(n-2)$ double points and $n-3$ other points chosen arbitrarily on the curve, a simply infinite set of curves of degree $n-2$, which we may suppose to have the equation

$$g(x, y) + t\,h(x, y) = 0,$$

where t is a variable parameter and $g=0$, $h=0$ are the equations of two particular members of the set. Any one of these curves meets the given curve in $n(n-2)$ points, of which $(n-1)(n-2)$ are accounted for by the $\frac{1}{2}(n-1)(n-2)$ double points, and $n-3$ by the other $n-3$ arbitrarily chosen points. These

$$(n-1)(n-2)+n-3=n(n-2)-1$$

points are independent of t; and so there is but *one* point of intersection which depends on t. The coordinates of this point are given by

$$g(x, y) + t\,h(x, y) = 0, \qquad f(x, y) = 0.$$

The elimination of y gives an equation of degree $n(n-2)$ in x, whose coefficients are polynomials in t; and but one root of this equation varies with t. The eliminant is therefore divisible by a factor of degree $n(n-2)-1$ which does not contain t. There remains a simple equation in x whose coefficients are polynomials in t. Thus the x-coordinate of the variable point is determined as a rational function of t, and the y-coordinate may be similarly determined.

We may therefore write

$$x = R_1(t), \qquad y = R_2(t).$$

If we reduce these fractions to the same denominator, we express the coordinates in the form

$$x = \frac{\phi_1(t)}{\phi_3(t)}, \qquad y = \frac{\phi_2(t)}{\phi_3(t)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where ϕ_1, ϕ_2, ϕ_3 are polynomials which have no common factor. The polynomials will in general be of degree n; none of them can be of

make. An ordinary multiple point of order k may be regarded as equivalent to $\frac{1}{2}k(k-1)$ ordinary double points. A curve of degree n which has an ordinary multiple point of order $n-1$, equivalent to $\frac{1}{2}(n-1)(n-2)$ ordinary double points, is therefore unicursal. The theory of higher plane curves abounds in puzzling particular cases which have to be fitted into the general theory by more or less obvious conventions, and to give a satisfactory account of a complicated compound singularity is sometimes by no means easy. In the investigation which follows we confine ourselves to the simplest case.

* Salmon, *l.c.*, p. 16.

higher degree, and one at least must be actually of that degree, since an arbitrary straight line

$$\lambda x + \mu y + \nu = 0$$

must cut the curve in exactly n points*.

We can now prove the second part of the theorem. If

$$x : y : 1 :: \phi_1(t) : \phi_2(t) : \phi_3(t),$$

where ϕ_1, ϕ_2, ϕ_3 are polynomials of degree n, then the line

$$ux + vy + w = 0$$

will meet the curve in n points whose parameters are given by

$$u\phi_1(t) + v\phi_2(t) + w\phi_3(t) = 0.$$

This equation will have a double root t_0 if

$$u\phi_1(t_0) + v\phi_2(t_0) + w\phi_3(t_0) = 0,$$
$$u\phi_1'(t_0) + v\phi_2'(t_0) + w\phi_3'(t_0) = 0.$$

Hence the equation of the tangent at the point t_0 is

$$\begin{vmatrix} x & y & 1 \\ \phi_1(t_0) & \phi_2(t_0) & \phi_3(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & \phi_3'(t_0) \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots(2).$$

If (x, y) is a fixed point, then the equation (2) may be regarded as an equation to determine the parameters of the points of contact of the tangents from (x, y). Now

$$\phi_2(t_0)\phi_3'(t_0) - \phi_2'(t_0)\phi_3(t_0)$$

is of degree $2n-2$ in t_0, the coefficient of t_0^{2n-1} obviously vanishing. Hence in general the number of tangents which can be drawn to a unicursal curve from a fixed point (the *class* of the curve) is $2n-2$. But the class of a curve whose only singular points are δ nodes is known† to be $n(n-1) - 2\delta$. Hence the number of nodes is

$$\tfrac{1}{2}\{n(n-1) - (2n-2)\} = \tfrac{1}{2}(n-1)(n-2).$$

It is perhaps worth pointing out how the proof which precedes requires modification if some only of the singular points are nodes and the rest ordinary cusps. The first part of the proof remains unaltered. The equation

* See Niewenglowski's *Cours de géométrie analytique*, vol. 2, p. 103. By way of illustration of the remark concerning particular cases in the footnote (§) to page 30, the reader may consider the example given by Niewenglowski in which

$$x = \frac{t^2}{t^2-1}, \quad y = \frac{t^2+1}{t^2-1};$$

equations which appear to represent the straight line $2x = y + 1$ (part of the line only, if we consider only real values of t).

† Salmon, *l.c.*, p. 54.

(2) must now be regarded as giving the values of t which correspond to (a) points at which the tangent passes through (x, y) and (b) cusps, since any line through a cusp 'cuts the curve in two coincident points'*. We have therefore

$$2n-2=m+\kappa,$$

where m is the class of the curve. But

$$m=n(n-1)-2\delta-3\kappa,\dagger$$

and so
$$\delta+\kappa=\tfrac{1}{2}(n-1)(n-2).\ddagger$$

8. (i) The preceding argument fails if $n<3$, but we have already seen that all conics are unicursal. The case next in importance is that of a cubic with a double point. If the double point is not at infinity we can, by a change of origin, reduce the equation of the curve to the form

$$(ax+by)(cx+dy)=px^3+3qx^2y+3rxy^2+sy^3;$$

and, by considering the intersections of the curve with the line $y=tx$, we find

$$x=\frac{(a+bt)(c+dt)}{p+3qt+3rt^2+st^3},\quad y=\frac{t(a+bt)(c+dt)}{p+3qt+3rt^2+st^3}.$$

If the double point is at infinity, the equation of the curve is of the form

$$(\alpha x+\beta y)^2(\gamma x+\delta y)+\epsilon x+\zeta y+\theta=0,$$

the curve having a pair of parallel asymptotes; and, by considering the intersection of the curve with the line $\alpha x+\beta y=t$, we find

$$x=-\frac{\delta t^3+\zeta t+\beta\theta}{(\beta\gamma-\alpha\delta)t^2+\epsilon\beta-\alpha\zeta},\quad y=\frac{\gamma t^3+\epsilon t+\alpha\theta}{(\beta\gamma-\alpha\delta)t^3+\epsilon\beta-\alpha\zeta}.$$

(ii) The case next in complexity is that of a quartic with three double points.

(a) The lemniscate $\quad(x^2+y^2)^2=a^2(x^2-y^2)$

has three double points, the origin and the circular points at infinity. The circle

$$x^2+y^2=t(x-y)$$

* This means of course that the equation obtained by substituting for x and y, in the equation of the line, their parametric expressions in terms of t, has a repeated root. This property is possessed by the tangent at an ordinary point and by any line through a cusp, but not by any line through a node except the two tangents.

† Salmon, *l.c.*, p. 65.

‡ I owe this remark to Mr A. B. Mayne. Dr Bromwich has however pointed out to me that substantially the same argument is given by Mr W. A. Houston, 'Note on unicursal plane curves', *Messenger of mathematics*, vol. 28, 1899, pp. 187–189.

passes through these points and one other fixed point at the origin, as it touches the curve there. Solving, we find

$$x=\frac{a^2t(t^2+a^2)}{t^4+a^4}, \qquad y=\frac{a^2t(t^2-a^2)}{t^4+a^4}.$$

(*b*) The curve $$2ay^3-3a^2y^2=x^4-2a^2x^2$$
has the double points (0, 0), (a, a), $(-a, a)$. Using the auxiliary conic

$$x^2-ay=tx(y-a),$$

we find $$x=\frac{a}{t^3}(2-3t^2), \qquad y=\frac{a}{2t^4}(2-3t^2)(2-t^2).$$

(iii) (*a*) The curve $$y^n=x^n+ax^{n-1}$$
has a multiple point of order $n-1$ at the origin, and is therefore unicursal. In this case it is sufficient to consider the intersection of the curve with the line $y=tx$. This may be harmonised with the general theory by regarding the curve

$$y^{n-3}(y-tx)=0,$$

as passing through each of the $\frac{1}{2}(n-1)(n-2)$ double points collected at the origin and through $n-3$ other fixed points collected at the point

$$x=-a, \quad y=0.$$

The curves $$y^n=x^n+ax^{n-1} \quad \text{.................................(1)},$$
$$y^n=1+az \quad \text{....................................(2)},$$
are projectively equivalent, as appears on rendering their equations homogeneous by the introduction of variables z in (1) and x in (2). We conclude that (2) is unicursal, having the maximum number of double points at infinity. In fact we may put

$$y=t, \quad az=t^n-1.$$

The integral $$\int R\{z, \sqrt[n]{(1+az)}\}\,dz$$
is accordingly an elementary function.

(*b*) The curve $$y^m=A(x-a)^\mu(x-b)^\nu$$
is unicursal if and only if either (i) $\mu=0$ or (ii) $\nu=0$ or (iii) $\mu+\nu=m$. Hence the integral

$$\int R\{x, (x-a)^{\mu/m}(x-b)^{\nu/m}\}\,dx$$

is an elementary function, for all forms of R, in these three cases only; of course it is integrable for special forms of R in other cases*.

* See Ptaszycki, 'Extrait d'une lettre adressée à M. Hermite', *Bulletin des sciences mathématiques*, ser. 2, vol. 12, 1888, pp. 262–270: Appell and Goursat, *Théorie des fonctions algébriques*, p. 245.

9. There is a similar theory connected with unicursal curves in space of any number of dimensions. Consider for example the integral

$$\int R\{x, \sqrt{(ax+b)}, \sqrt{(cx+d)}\}\,dx.$$

A linear substitution $x = lx + m$ reduces this integral to the form

$$\int R_1\{y, \sqrt{(y+2)}, \sqrt{(y-2)}\}\,dy;$$

and this integral can be rationalised by putting

$$y = t^2 + \frac{1}{t^2}, \quad \sqrt{(y+2)} = t + \frac{1}{t}, \quad \sqrt{(y-2)} = t - \frac{1}{t}.$$

The curve whose Cartesian coordinates ξ, η, ζ are given by

$$\xi : \eta : \zeta : 1 :: t^4 + 1 : t(t^2+1) : t(t^2-1) : t^2,$$

is a unicursal twisted quartic, the intersection of the parabolic cylinders

$$\xi = \eta^2 - 2, \quad \xi = \zeta^2 + 2.$$

It is easy to deduce that the integral

$$\int R\left\{x, \sqrt{\left(\frac{ax+b}{mx+n}\right)}, \sqrt{\left(\frac{cx+d}{mx+n}\right)}\right\} dx$$

is always an elementary function.

10. When the deficiency of the curve $f(x, y) = 0$ is not zero, the integral

$$\int R(x, y)\,dx$$

is in general not an elementary function; and the consideration of such integrals has consequently introduced a whole series of classes of new transcendents into analysis. The simplest case is that in which the deficiency is unity: in this case, as we shall see later on, the integrals are expressible in terms of elementary functions and certain new transcendents known as elliptic integrals. When the deficiency rises above unity the integration necessitates the introduction of new transcendents of growing complexity.

But there are infinitely many particular cases in which integrals, associated with curves whose deficiency is unity or greater than unity, can be expressed in terms of elementary functions, or are even algebraical themselves. For instance the deficiency of

$$y^2 = 1 + x^3$$

is unity. But

$$\int \frac{x+1}{x-2} \frac{dx}{\sqrt{(1+x^3)}} = 3 \log \frac{(1+x)^2 - 3\sqrt{(1+x^3)}}{(1+x)^2 + 3\sqrt{(1+x^3)}},$$

$$\int \frac{2-x^3}{1+x^3} \frac{dx}{\sqrt{(1+x^3)}} = \frac{2x}{\sqrt{(1+x^3)}}.$$

And, before we say anything concerning the new transcendents to which integrals of this class in general give rise, we shall consider what has been done in the way of formulating rules to enable us to identify such cases and to assign the form of the integral when it is an elementary function. It will be as well to say at once that this problem has not been solved completely.

11. The first general theorem of this character deals with the case in which the integral is algebraical, and asserts that *if*

$$u = \int y\,dx$$

is an algebraical function of x, then it is a rational function of x and y.

Our proof will be based on the following lemmas.

(1) *If $f(x, y)$ and $g(x, y)$ are polynomials, and there is no factor common to all the coefficients of the various powers of y in $g(x, y)$; and*

$$f(x, y) = g(x, y)\, h(x),$$

where $h(x)$ is a rational function of x; then $h(x)$ is a polynomial.

Let $h = P/Q$, where P and Q are polynomials without a common factor. Then

$$fQ = gP.$$

If $x - a$ is a factor of Q, then

$$g(a, y) = 0$$

for all values of y; and so all the coefficients of powers of y in $g(x, y)$ are divisible by $x - a$, which is contrary to our hypotheses. Hence Q is a constant and h a polynomial.

(2) *Suppose that $f(x, y)$ is an irreducible polynomial, and that $y_1, y_2, \dots, y_n$ are the roots of*

$$f(x, y) = 0$$

in a certain domain D. Suppose further that $\phi(x, y)$ is another polynomial, and that

$$\phi(x, y_1) = 0.$$

Then $$\phi(x, y_s) = 0,$$

where y_s is any one of the roots of (1) *; and*

$$\phi(x, y) = f(x, y)\,\psi(x, y),$$

where $\psi(x, y)$ also is a polynomial in x and y.

Let us determine the highest common factor ϖ of f and ϕ, considered as polynomials in y, by the ordinary process for the determination of the highest common factor of two polynomials. This process depends only on a series of algebraical divisions, and so ϖ is a polynomial in y with coefficients rational in x. We have therefore

$$\varpi(x, y) = \omega(x, y)\,\lambda(x) \quad \text{.....................(1)},$$

$$f(x, y) = \omega(x, y)\,p(x, y)\,\mu(x) = g(x, y)\,\mu(x) \quad \text{......(2)},$$

$$\phi(x, y) = \omega(x, y)\,q(x, y)\,\nu(x) = h(x, y)\,\nu(x) \quad \text{......(3)},$$

where ω, p, q, g, and h are polynomials and λ, μ, and ν rational functions; and evidently we may suppose that neither in g nor in h have the coefficients of all powers of y a common factor. Hence, by Lemma (1), μ and ν are polynomials. But f is irreducible, and therefore μ and either ω or p must be constants. If ω were a constant, ϖ would be a function of x only. But this is impossible. For we can determine polynomials L, M in y, with coefficients rational in x, such that

$$Lf + M\phi = \varpi \quad \text{..........................(4)},$$

and the left-hand side of (4) vanishes when we write y_1 for y. Hence p is a constant, and so ω is a constant multiple of f. The truth of the lemma now follows from (3).

It follows from Lemma (2) that *y cannot satisfy any equation of degree less than n whose coefficients are polynomials in x.*

(3) *If y is an algebraical function of x, defined by an equation*

$$f(x, y) = 0 \quad \text{.............................(1)}$$

of degree n, then any rational function $R(x, y)$ of x and y can be expressed in the form

$$R(x, y) = R_0 + R_1 y + \dots + R_{n-1} y^{n-1} \quad \text{..............(2)},$$

where R_0, R_1, ..., R_{n-1} are rational functions of x.

The function y is one of the n roots of (1). Let $y, y', y'', \dots$ be the complete system of roots. Then

$$R(x,y) = \frac{P(x,y)}{Q(x,y)}$$
$$= \frac{P(x,y)\,Q(x,y')\,Q(x,y'')\dots}{Q(x,y)\,Q(x,y')\,Q(x,y'')\dots} \quad \dots\dots\dots\dots\dots(3),$$

where P and Q are polynomials. The denominator is a polynomial in x whose coefficients are symmetric polynomials in $y, y', y'', \dots$, and is therefore, by II., § 3, (i), a rational function of x. On the other hand

$$Q(x,y')\,Q(x,y'')\dots$$

is a polynomial in x whose coefficients are symmetric polynomials in $y', y'', \dots$, and therefore, by II., § 3, (ii), polynomials in y with coefficients rational in x. Thus the numerator of (3) is a polynomial in y with coefficients rational in x.

It follows that $R(x,y)$ is a polynomial in y with coefficients rational in x. From this polynomial we can eliminate, by means of (1), all powers of y as high as or higher than the nth. Hence $R(x,y)$ is of the form prescribed by the lemma.

12. We proceed now to the proof of our main theorem. We have

$$\int y\,dx = u$$

where u is algebraical. Let

$$f(x,y) = 0, \quad \psi(x,u) = 0 \quad \dots\dots\dots\dots\dots\dots(1)$$

be the irreducible equations satisfied by y and u, and let us suppose that they are of degrees n and m respectively. The first stage in the proof consists in showing that

$$m = n.$$

It will be convenient now to write y_1, u_1 for y, u, and to denote by

$$y_1, y_2, \dots, y_n, \qquad u_1, u_2, \dots, u_m,$$

the complete systems of roots of the equations (1).

We have $$\psi(x, u_1) = 0,$$

and so $$\chi_1 = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial u_1}\frac{du_1}{dx} = \frac{\partial\psi}{\partial x} + y_1\frac{\partial\psi}{\partial u_1} = 0.$$

Now let $$\Omega(x, u_1) = \prod_{r=1}^{n}\left(\frac{\partial\psi}{\partial x} + y_r\frac{\partial\psi}{\partial u_1}\right).$$

Then Ω is a polynomial in u_1, with coefficients symmetric in $y_1, y_2, \dots, y_n$ and therefore rational in x.

The equations $\psi = 0$ and $\Omega = 0$ have a root u_1 in common, and the first equation is irreducible. It follows, by Lemma (2) of § 11, that

$$\Omega(x, u_s) = 0$$

for $s = 1, 2, \dots, m$.* And from this it follows that, when s is given, we have

$$\frac{\partial \psi}{\partial x} + y_r \frac{\partial \psi}{\partial u_s} = 0 \quad \text{.............................(2)}$$

for some value of the suffix r.

But we have also

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial u_s}\frac{du_s}{dx} = 0 \quad \text{..........................(3)};$$

and from (2) and (3) it follows † that

$$\frac{du_s}{dx} = y_r \quad \text{..............................(4)},$$

i.e. that *every u is the integral of some y.*

In the same way we can show that *every y is the derivative of some u.* Let

$$\omega(x, y_1) = \prod_{s=1}^{m} \left(\frac{\partial \psi}{\partial x} + y_1 \frac{\partial \psi}{\partial u_s}\right).$$

Then ω is a polynomial in y_1, with coefficients symmetric in $u_1, u_2, \dots, u_m$ and therefore rational in x. The equations $f = 0$ and $\omega = 0$ have a root y_1 in common, and so

$$\omega(x, y_r) = 0$$

for $r = 1, 2, \dots, n$. From this we deduce that, when r is given, (2) must be true for some value of s, and so that the same is true of (4).

Now it is impossible that, in (4), two different values of s should correspond to the same value of r. For this would involve

$$u_s - u_t = c$$

where $s \neq t$ and c is a constant. Hence we should have

$$\psi(x, u_s) = 0, \quad \psi(x, u_s - c) = 0.$$

* If $p(x)$ is the least common multiple of the denominators of the coefficients of powers of u in Ω, then

$$\Omega(x, u)\, p(x) = \chi(x, u),$$

where χ is a polynomial. Applying Lemma (2), we see that $\chi(x, u_s) = 0$, and so

$$\Omega(x, u_s) = 0.$$

† It is impossible that ψ and $\frac{\partial \psi}{\partial u}$ should both vanish for $u = u_s$, since ψ is irreducible.

Subtracting these equations, we should obtain an equation of degree $m-1$ in u_s, with coefficients which are polynomials in x; and this is impossible. In the same way we can prove that two different values of r cannot correspond to the same value of s.

The equation (4) therefore establishes a one-one correspondence between the values of r and s. It follows that

$$m = n.$$

It is moreover evident that, by arranging the suffixes properly, we can make

$$\frac{du_r}{dx} = y_r \qquad \text{(5)}$$

for $r = 1, 2, \ldots, n$.

13. We have

$$y_r = \frac{du_r}{dx} = -\frac{\partial\psi}{\partial x}\bigg/\frac{\partial\psi}{\partial u_r} = R(x, u_r),$$

where R is a rational function which may, in virtue of Lemma (3) of § 11, be expressed as a polynomial of degree $n-1$ in u_r, with coefficients rational in x.

The product

$$\prod_{s \neq r} (z - y_s)$$

is a polynomial of degree $n-1$ in z, with coefficients which are symmetric polynomials in $y_1, y_2, \ldots, y_{r-1}, y_{r+1}, \ldots, y_n$ and therefore, by II., § 3, (ii), polynomials in y_r with coefficients rational in x. Replacing y_r by its expression as a polynomial in u_r obtained above, and eliminating u_r^n and all higher powers of u_r, we obtain an equation

$$\prod_{s \neq r} (z - y_s) = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} S_{j,k}(x)\, z^j u_r^k,$$

where the S's are rational functions of x which are, from the method of their formation, independent of the particular value of r selected. We may therefore write

$$\prod_{s \neq r} (z - y_s) = P(x, z, u_r),$$

where P is a polynomial in z and u_r with coefficients rational in x. It is evident that

$$P(x, y_s, u_r) = 0$$

for every value of s other than r. In particular

$$P(x, y_1, u_r) = 0 \qquad (r = 2, 3, \ldots, n).$$

It follows that the $n-1$ roots of the equation in u

$$P(x, y_1, u) = 0$$

are $u_2, u_3, \ldots, u_n$. We have therefore

$$P(x, y_1, u) = T_0(x, y_1) \prod_2^n (u - u_r)$$

$$= T_0(x, y_1)\{u^{n-1} - u^{n-2}(u_2 + u_3 + \ldots + u_n) + \ldots\}$$

$$= T_0(x, y_1)\left[u^{n-1} + u^{n-2}\left\{u_1 + \frac{B_1(x)}{B_0(x)}\right\} + \ldots\right],$$

where $T_0(x, y_1)$ is the coefficient of u^{n-1} in P, and $B_0(x)$ and $B_1(x)$ are the coefficients of u^n and u^{n-1} in ψ. Equating the coefficients of u^{n-2} on the two sides of this equation, we obtain

$$u_1 + \frac{B_1(x)}{B_0(x)} = \frac{T_1(x, y_1)}{T_0(x, y_1)},$$

where $T_1(x, y_1)$ is the coefficient of u^{n-2} in P. Thus the theorem is proved.

14. We can now apply Lemma (3) of § 11; and we arrive at the final conclusion that *if*

$$\int y\,dx$$

is algebraical then it can be expressed in the form

$$R_0 + R_1 y + \ldots + R_{n-1} y^{n-1},$$

where R_0, R_1, ... are rational functions of x.

The most important case is that in which

$$y = \sqrt[n]{\{R(x)\}},$$

where $R(x)$ is rational. In this case

$$y^n = R(x) \quad\ldots\ldots\ldots\ldots(1),$$

$$\frac{dy}{dx} = \frac{R'(x)}{ny^{n-1}} \quad\ldots\ldots\ldots\ldots(2).$$

But

$$y = R_0' + R_1' y + \ldots + R'_{n-1} y^{n-1}$$
$$+ \{R_1 + 2R_2 y + \ldots + (n-1)R_{n-1} y^{n-2}\}\frac{dy}{dx} \quad\ldots\ldots\ldots(3).$$

Eliminating $\frac{dy}{dx}$ between these equations, we obtain an equation

$$\varpi(x, y) = 0 \quad\ldots\ldots\ldots\ldots(4),$$

where $\varpi(x, y)$ is a polynomial. It follows from Lemma (2) of § 11 that this equation must be satisfied by all the roots of (1). Thus (4) is still true if we replace y by any other root y' of (1); and as

(2) is still true when we effect this substitution, it follows that (3) is also still true. Integrating, we see that the equation

$$\int y\,dx = R_0 + R_1 y + \dots + R_{n-1} y^{n-1}$$

is true when y is replaced by y'. We may therefore replace y by ωy, ω being any primitive nth root of unity. Making this substitution, and multiplying by ω^{n-1}, we obtain

$$\int y\,dx = \omega^{n-1} R_0 + R_1 y + \omega R_2 y + \dots + \omega^{n-2} R_{n-1} y^{n-1};$$

and on adding the n equations of this type we obtain

$$\int y\,dx = R_1 y.$$

Thus in this case the functions R_0, R_2, ..., R_{n-1} all disappear.

It has been shown by Liouville* that the preceding results enable us to obtain in all cases, by a finite number of elementary algebraical operations, a solution of the problem '*to determine whether* $\int y\,dx$ *is algebraical, and to find the integral when it is algebraical*'.

15. It would take too long to attempt to trace in detail the steps of the general argument. We shall confine ourselves to a solution of a particular problem which will give a sufficient illustration of the general nature of the arguments which must be employed.

We shall determine under what circumstances the integral

$$\int \frac{dx}{(x-p)\sqrt{(ax^2+2bx+c)}}$$

is algebraical. This question might of course be answered by actually evaluating the integral in the general case and finding when the integral function reduces to an algebraical function. We are now, however, in a position to answer it without any such integration.

We shall suppose first that $ax^2+2bx+c$ is not a perfect square. In this case

$$y = \frac{1}{\sqrt{X}},$$

where

$$X = (x-p)^2(ax^2+2bx+c),$$

and if $\int y\,dx$ is algebraical it must be of the form

$$\frac{R(x)}{\sqrt{X}}.$$

Hence

$$y = \frac{d}{dx}\left(\frac{R}{\sqrt{X}}\right),$$

or

$$2X = 2XR' - RX'.$$

* 'Premier mémoire sur la détermination des intégrales dont la valeur est algébrique', *Journal de l'École Polytechnique*, vol. 14, cahier 22, 1833, pp. 124–148; 'Second mémoire...', *ibid.*, pp. 149–193.

We can now show that R is a polynomial in x. For if $R=U/V$, where U and V are polynomials, then V, if not a mere constant, must contain a factor

$$(x-a)^{\mu} \qquad (\mu>0),$$

and we can put
$$R=\frac{U}{W(x-a)^{\mu}},$$
where U and W do not contain the factor $x-a$. Substituting this expression for R, and reducing, we obtain

$$\frac{2\mu UWX}{x-a}=2U'WX-2UW'X-UWX'-2W^2X(x-a)^{\mu}.$$

Hence X must be divisible by $x-a$. Suppose then that

$$X=(x-a)^k Y,$$

where Y is prime to $x-a$. Substituting in the equation last obtained we deduce

$$\frac{(2\mu+k)UWY}{x-a}=2U'WY-2UW'Y-UWY'-2W^2Y(x-a)^{\mu},$$

which is obviously impossible, since neither U, W, nor Y is divisible by $x-a$. Thus V must be a constant. Hence

$$\int\frac{dx}{(x-p)\sqrt{(ax^2+2bx+c)}}=\frac{U(x)}{(x-p)\sqrt{(ax^2+2bx+c)}},$$

where $U(x)$ is a polynomial.

Differentiating and clearing of radicals we obtain

$$\{(x-p)(U'-1)-U\}(ax^2+2bx+c)=U(x-p)(ax+b).$$

Suppose that the first term in U is Ax^m. Equating the coefficients of x^{m+2}, we find at once that $m=2$. We may therefore take

$$U=Ax^2+2Bx+C,$$

so that

$$\{(x-p)(2Ax+2B-1)-Ax^2-2Bx-C\}(ax^2+2bx+c)$$
$$=(x-p)(ax+b)(Ax^2+2Bx+C)\ldots\ldots(1).$$

From (1) it follows that

$$(x-p)(ax+b)(Ax^2+2Bx+C)$$

is divisible by $ax^2+2bx+c$. But $ax+b$ is not a factor of $ax^2+2bx+c$, as the latter is not a perfect square. Hence either (i) $ax^2+2bx+c$ and $Ax^2+2Bx+C$ differ only by a constant factor or (ii) the two quadratics have one and only one factor in common, and $x-p$ is also a factor of $ax^2+2bx+c$. In the latter case we may write

$$ax^2+2bx+c=a(x-p)(x-q), \quad Ax^2+2Bx+C=A(x-q)(x-r),$$

where $p\neq q$, $p\neq r$. It then follows from (1) that

$$a(x-p)(2Ax+2B-1)-aA(x-q)(x-r)=A(ax+b)(x-r).$$

Hence $2Ax+2B-1$ is divisible by $x-r$. Dividing by $aA(x-r)$ we obtain

$$2(x-p)-(x-q)=x+\frac{b}{a}=x-\tfrac{1}{2}(p+q),$$

and so $p=q$, which is untrue.

Hence case (ii) is impossible, and so $ax^2+2bx+c$ and $Ax^2+2Bx+C$ differ only by a constant factor. It then follows from (1) that $x-p$ is a factor of $ax^2+2bx+c$; and the result becomes

$$\int\frac{dx}{(x-p)\sqrt{(ax^2+2bx+c)}}=K\frac{\sqrt{(ax^2+2bx+c)}}{x-p},$$

where K is a constant. It is easily verified that this equation is actually true when $ap^2+2bp+c=0$, and that

$$K=\frac{1}{\sqrt{(b^2-ac)}}.$$

The formula is equivalent to

$$\int\frac{dx}{(x-p)\sqrt{\{(x-p)(x-q)\}}}=\frac{2}{q-p}\sqrt{\left(\frac{x-q}{x-p}\right)}.$$

There remains for consideration the case in which $ax^2+2bx+c$ is a perfect square, say $a(x-q)^2$. Then

$$\int\frac{dx}{(x-p)(x-q)}$$

must be rational, and so $p=q$.

As a further example, the reader may verify that if

$$y^3-3y+2x=0$$

then

$$\int y\,dx=\frac{3}{8}(2xy-y^2).*$$

16. The theorem of § 11 enables us to complete the proof of the two fundamental theorems stated without proof in II., § 5, viz.

(a) e^x is not an algebraical function of x,

(b) $\log x$ is not an algebraical function of x.

We shall prove (b) as a special case of a more general theorem, viz. '*no sum of the form*

$$A\log(x-\alpha)+B\log(x-\beta)+\ldots,$$

in which the coefficients A, B, ... are not all zero, can be an algebraical function of x'. To prove this we have only to observe that the sum in question is the integral of a rational function of x. If then it is algebraical it must, by the theorem of § 11, be rational, and this we have already seen to be impossible (IV., 2).

That e^x is not algebraical now follows at once from the fact that it is the inverse function of $\log x$.

17. The general theorem of § 11 gives the first step in the rigid proof of 'Laplace's principle' stated in III., § 2. On account of the immense importance of this principle we repeat Laplace's words:

* Raffy, 'Sur les quadratures algébriques et logarithmiques', *Annales de l'École Normale*, ser. 3, vol. 2, 1885, pp. 185–206.

'*l'intégrale d'une fonction différentielle ne peut contenir d'autres quantités radicaux que celles qui entrent dans cette fonction*'. This general principle, combined with arguments similar to those used above (§ 15) in a particular case, enables us to prove without difficulty that a great many integrals cannot be algebraical, notably the standard elliptic integrals

$$\int \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}}, \qquad \int \sqrt{\left(\frac{1-x^2}{1-k^2x^2}\right)}\, dx, \qquad \int \frac{dx}{\sqrt{(4x^3-g_2x-g_3)}}$$

which give rise by inversion to the elliptic functions.

18. We must now consider in a very summary manner the more difficult question of the nature of those integrals of algebraical functions which are expressible in finite terms by means of the elementary transcendental functions. In the first place *no integral of any algebraical function can contain any exponential.* Of this theorem it is, as we remarked before, easy to become convinced by a little reflection, as doubtless did Laplace, who certainly possessed no rigorous proof. The reader will find little difficulty in coming to the conclusion that exponentials cannot be eliminated from an elementary function by differentiation. But we would strongly recommend him to study the exceedingly beautiful and ingenious proof of this proposition given by Liouville*. We have unfortunately no space to insert it here.

It is instructive to consider particular cases of this theorem. Suppose for example that $\int y\,dx$, where y is algebraical, were a polynomial in x and e^x, say

$$\Sigma\Sigma a_{m,n} x^m e^{nx} \quad \text{..............................(1).}$$

When this expression is differentiated, e^x must disappear from it: otherwise we should have an algebraical relation between x and e^x. Expressing the conditions that the coefficient of every power of e^x in the differential coefficient of (1) vanishes identically, we find that the same must be true of (1), so that after all the integral does not really contain e^x. Liouville's proof is in reality a development of this idea.

The integral of an algebraical function, if expressible in terms of elementary functions, can therefore only contain algebraical or logarithmic functions. The next step is to show that the logarithms must be simple logarithms of algebraical functions and can only enter linearly, so that the general integral must be of the type

$$\int y\,dx = u + A \log v + B \log w + \dots,$$

* 'Mémoire sur les transcendantes elliptiques considérées comme fonctions de leur amplitude', *Journal de l'École Polytechnique*, vol. 14, cahier 23, 1834, pp. 37–83. The proof may also be found in Bertrand's *Calcul intégral*, p. 99.

where A, B, ... are constants and u, v, w, ... algebraical functions. Only when the logarithms occur in this simple form will differentiation eliminate them.

Lastly it can be shown by arguments similar to those of §§ 11–14 that u, v, w, ... are rational functions of x and y. Thus $\int y\,dx$, if an elementary function, is *the sum of a rational function of x and y and of certain constant multiples of logarithms of such functions.* We can suppose that no two of A, B, ... are commensurable, or indeed, more generally, that no linear relation

$$A\alpha + B\beta + \ldots = 0,$$

with rational coefficients, holds between them. For if such a relation held then we could eliminate A from the integral, writing it in the form

$$\int y\,dx = u + B\log(wv^{-\beta/\alpha}) + \ldots.$$

It is instructive to verify the truth of this theorem in the special case in which the curve $f(x, y)=0$ is unicursal. In this case x and y are rational functions $R(t)$, $S(t)$ of a parameter t, and the integral, being the integral of a rational function of t, is of the form

$$u + A\log v + B\log w + \ldots,$$

where u, v, w, ... are rational functions of t. But t may be expressed, by means of elementary algebraical operations, as a rational function of x and y. Thus u, v, w, ... are rational functions of x and y.

The case of greatest interest is that in which y is a rational function of x and $\sqrt{X}$, where X is a polynomial. As we have already seen, y can in this case be expressed in the form

$$P + \frac{Q}{\sqrt{X}},$$

where P and Q are rational functions of x. We shall suppress the rational part and suppose that $y = Q/\sqrt{X}$. In this case the general theorem gives

$$\int\frac{Q}{\sqrt{X}}\,dx = S + \frac{T}{\sqrt{X}} + A\log(\alpha + \beta\sqrt{X}) + B\log(\gamma + \delta\sqrt{X}) + \ldots,$$

where S, T, α, β, γ, δ, ... are rational. If we differentiate this equation we obtain an algebraical identity in which we can change the sign of $\sqrt{X}$. Thus we may change the sign of $\sqrt{X}$ in the integral equation. If we do this and subtract, and write $2A$, ... for A, ..., we obtain

$$\int\frac{Q}{\sqrt{X}}\,dx = \frac{T}{\sqrt{X}} + A\log\frac{\alpha + \beta\sqrt{X}}{\alpha - \beta\sqrt{X}} + B\log\frac{\gamma + \delta\sqrt{X}}{\gamma - \delta\sqrt{X}} + \ldots,$$

which is the standard form for such an integral. It is evident that we may suppose $\alpha, \beta, \gamma, \ldots$ to be polynomials.

19. (i) By means of this theorem it is possible to prove that a number of important integrals, and notably the integrals

$$\int \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}}, \quad \int \sqrt{\left\{\frac{1-x^2}{1-k^2x^2}\right\}}\, dx, \quad \int \frac{dx}{\sqrt{(4x^3-g_2x-g_3)}},$$

are not expressible in terms of elementary functions, and so represent genuinely new transcendents. The formal proof of this was worked out by Liouville*; it rests merely on a consideration of the possible forms of the differential coefficients of expressions of the form

$$\frac{T}{\sqrt{X}} + A \log \frac{\alpha+\beta\sqrt{X}}{\alpha-\beta\sqrt{X}} + \ldots,$$

and the arguments used are purely algebraical and of no great theoretical difficulty. The proof is however too detailed to be inserted here. It is not difficult to find shorter proofs, but these are of a less elementary character, being based on ideas drawn from the theory of functions†.

The general questions of this nature which arise in connection with integrals of the form

$$\int \frac{Q}{\sqrt{X}}\, dx,$$

or, more generally,

$$\int \frac{Q}{\sqrt[m]{X}}\, dx,$$

are of extreme interest and difficulty. The case which has received most attention is that in which $m=2$ and X is of the third or fourth degree, in which case the integral is said to be *elliptic*. An integral of this kind is called *pseudo-elliptic* if it is expressible in terms of algebraical and logarithmic functions. Two examples were given above (§ 10). General methods have been given for the construction of such integrals, and it has been shown that certain interesting forms are pseudo-elliptic. In Goursat's *Cours d'analyse* ‡, for instance, it is shown that if $f(x)$ is a rational function such that

$$f(x) + f\left(\frac{1}{k^2x}\right) = 0,$$

then

$$\int \frac{f(x)\, dx}{\sqrt{\{x(1-x)(1-k^2x)\}}}$$

is pseudo-elliptic. But no method has been devised as yet by which we can always determine in a finite number of steps whether a *given* elliptic integral

* See Liouville's memoir quoted on p. 45 (pp. 45 *et seq.*).

† The proof given by Laurent (*Traité d'analyse*, vol. 4, pp. 153 *et seq.*) appears at first sight to combine the advantages of both methods of proof, but unfortunately will not bear a closer examination.

‡ Second edition, vol. 1, pp. 267–269.

is pseudo-elliptic, and integrate it if it is, and there is reason to suppose that no such method can be given. And up to the present it has not, so far as we know, been proved rigorously and explicitly that (*e.g.*) the function

$$u=\int\frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}}$$

is not a root of an elementary transcendental equation; all that has been shown is that it is not *explicitly* expressible in terms of elementary transcendents. The processes of reasoning employed here, and in the memoirs to which we have referred, do not therefore suffice to prove that the inverse function $x=\operatorname{sn} u$ is not an elementary function of u. Such a proof must rest on the known properties of the function $\operatorname{sn} u$, and would lie altogether outside the province of this tract.

The reader who desires to pursue the subject further will find references to the original authorities in Appendix I.

(ii) One particular class of integrals which is of especial interest is that of the *binomial integrals*

$$\int x^m (ax^n+b)^p\,dx,$$

where m, n, p are rational. Putting $ax^n=bt$, and neglecting a constant factor, we obtain an integral of the form

$$\int t^q (1+t)^p\,dt,$$

where p and q are rational. If p is an integer, and q a fraction r/s, this integral can be evaluated at once by putting $t=u^s$, a substitution which rationalises the integrand. If q is an integer, and $p=r/s$, we put $1+t=u^s$. If $p+q$ is an integer, and $p=r/s$, we put $1+t=tu^s$.

It follows from Tschebyschef's researches (to which references are given in Appendix I) that these three cases are the only ones in which the integral can be evaluated in finite form.

20. In §§ 7–9 we considered in some detail the integrals connected with curves whose deficiency is zero. We shall now consider in a more summary way the case next in simplicity, that in which the deficiency is unity, so that the number of double points is

$$\tfrac{1}{2}(n-1)(n-2)-1=\tfrac{1}{2}n(n-3).$$

It has been shown by Clebsch* that in this case the coordinates of the points of the curve can be expressed as *rational functions of a parameter t and of the square root of a polynomial in t of the third or fourth degree.*

* 'Über diejenigen Curven, deren Coordinaten sich als elliptische Functionen eines Parameters darstellen lassen', *Journal für Mathematik*, vol. 64, 1865, pp. 210–270.

The fact is that the curves

$$y^2 = a + bx + cx^2 + dx^3,$$
$$y^2 = a + bx + cx^2 + dx^3 + ex^4,$$

are the simplest curves of deficiency 1. The first is the typical cubic without a double point. The second is a quartic with two double points, in this case coinciding in a 'tacnode' at infinity, as we see by making the equation homogeneous with z, writing 1 for y, and then comparing the resulting equation with the form treated by Salmon on p. 215 of his *Higher plane curves*. The reader who is familiar with the theory of algebraical plane curves will remember that the deficiency of a curve is unaltered by any birational transformation of coordinates, and that any curve can be birationally transformed into any other curve of the same deficiency, so that any curve of deficiency 1 can be birationally transformed into the cubic whose equation is written above.

The argument by which this general theorem is proved is very much like that by which we proved the corresponding theorem for unicursal curves. The simplest case is that of the general cubic curve. We take a point on the curve as origin, so that the equation of the curve is of the form

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 + ex^2 + 2fxy + gy^2 + hx + ky = 0.$$

Let us consider the intersections of this curve with the secant $y = tx$. Eliminating y, and solving the resulting quadratic in x, we see that the only irrationality which enters into the expression of x is

$$\sqrt{(T_2^2 - 4T_1T_3)},$$

where $\quad T_1 = h + kt, \quad T_2 = e + 2ft + gt^2, \quad T_3 = a + 3bt + 3ct^2 + dt^3.$

A more elegant method has been given by Clebsch*. If we write the cubic in the form

$$LMN = P,$$

where L, M, N, P are linear functions of x and y, so that L, M, N are the asymptotes, then the hyperbolas $LM = t$ will meet the cubic in four fixed points at infinity, and therefore in two points only which depend on t. For these points

$$LM = t, \quad P = tN.$$

Eliminating y from these equations, we obtain an equation of the form

$$Ax^2 + 2Bx + C = 0,$$

where A, B, C are quadratics in t. Hence

$$x = -\frac{B}{A} \pm \frac{\sqrt{(B^2 - AC)}}{A} = R(t, \sqrt{T}),$$

* See Hermite, *Cours d'analyse*, pp. 422–425.

where $T = B^2 - AC$ is a polynomial in t of degree not higher than the fourth.

Thus if the curve is

$$x^3 + y^3 - 3axy + 1 = 0,$$

so that

$$L = \omega x + \omega^2 y + a, \quad M = \omega^2 x + \omega y + a, \quad N = x + y + a, \quad P = a^3 - 1,$$

ω being an imaginary cube root of unity, then we find that the line

$$x + y + a = \frac{a^3 - 1}{t}$$

meets the curve in the points given by

$$x = \frac{b - at}{2t} \pm \frac{\sqrt{(3T)}}{6t}, \qquad y = \frac{b - at}{2t} \mp \frac{\sqrt{(3T)}}{6t},$$

where $b = a^3 - 1$ and

$$T = 4t^3 - 9a^2t^2 + 6abt - b^2.$$

In particular, for the curve

$$x^3 + y^3 + 1 = 0,$$

we have

$$x = \frac{-\sqrt{3} + \sqrt{(4t^3 - 1)}}{2t\sqrt{3}}, \qquad y = \frac{-\sqrt{3} - \sqrt{(4t^3 - 1)}}{2t\sqrt{3}}.$$

21. It will be plain from what precedes that

$$\int R\{x, \sqrt[3]{(a + bx + cx^2 + dx^3)}\}\, dx$$

can always be reduced to an elliptic integral, the deficiency of the cubic

$$y^3 = a + bx + cx^2 + dx^3$$

being unity.

In general integrals associated with curves whose deficiency is greater than unity cannot be so reduced. But associated with every curve of, let us say, deficiency 2 there will be an infinity of integrals

$$\int R(x, y)\, dx$$

reducible to elliptic integrals or even to elementary functions; and there are curves of deficiency 2 for which *all* such integrals are reducible.

For example, the integral

$$\int R\{x, \sqrt{(x^6 + ax^4 + bx^2 + c)}\}\, dx$$

may be split up into the sum of the integral of a rational function and two integrals of the types

$$\int \frac{R(x^2)\,dx}{\sqrt{(x^6 + ax^4 + bx^2 + c)}}, \qquad \int \frac{xR(x^2)\,dx}{\sqrt{(x^6 + ax^4 + bx^2 + c)}},$$

and each of these integrals becomes elliptic on putting $x^2 = t$. But the deficiency of

$$y^2 = x^6 + ax^4 + bx^2 + c$$

is 2. Another example is given by the integral

$$\int R\{x, \sqrt[3]{(x^4 + ax^3 + bx^2 + cx + d)}\}\,dx.^*$$

22. It would be beside our present purpose to enter into any details as to the general theory of elliptic integrals, still less of the integrals (usually called Abelian) associated with curves of deficiency greater than unity. We have seen that if the deficiency is unity then the integral can be transformed into the form

$$\int R(x, \sqrt{X})\,dx$$

where

$$X = x^4 + ax^3 + bx^2 + cx + d.\dagger$$

It can be shown that, by a transformation of the type

$$x = \frac{\alpha t + \beta}{\gamma t + \delta},$$

this integral can be transformed into an integral

$$\int R(t, \sqrt{T})\,dt$$

where

$$T = t^4 + At^2 + B.$$

We can then, as when T is of the second degree (§ 3), decompose this integral into two integrals of the forms

$$\int R(t)\,dt, \qquad \int \frac{R(t)\,dt}{\sqrt{T}}.$$

Of these integrals the first is elementary, and the second can be

* See Legendre, *Traité des fonctions elliptiques*, vol. 1, chs. 26–27, 32–33; Bertrand, *Calcul intégral*, pp. 67 *et seq.*; and Enneper, *Elliptische Funktionen*, note 1, where abundant references are given.

† There is a similar theory for curves of deficiency 2, in which X is of the *sixth* degree.

decomposed* into the sum of an algebraical term, of certain multiples of the integrals

$$\int \frac{dt}{\sqrt{T}}, \quad \int \frac{t^2 dt}{\sqrt{T}},$$

and of a number of integrals of the type

$$\int \frac{dt}{(t-\tau)\sqrt{T}}.$$

These integrals cannot in general be reduced to elementary functions, and are therefore new transcendents.

We will only add, before leaving this part of our subject, that the algebraical part of these integrals can be found by means of the elementary algebraical operations, as was the case with the rational part of the integral of a rational function, and with the algebraical part of the simple integrals considered in §§ 14–15.

VI. Transcendental functions

1. The theory of the integration of transcendental functions is naturally much less complete than that of the integration of rational or even of algebraical functions. It is obvious from the nature of the case that this must be so, as there is no general theorem concerning transcendental functions which in any way corresponds to the theorem that any algebraical combination of algebraical functions may be regarded as a simple algebraical function, the root of an equation of a simple standard type.

It is indeed almost true to say that there is no general theory, or that the theory reduces to an enumeration of the few cases in which the integral may be transformed by an appropriate substitution into an integral of a rational or algebraical function. These few cases are however of great importance in applications.

2. (i) The integral

$$\int F(e^{ax}, e^{bx}, \ldots, e^{kx})\, dx$$

where F is an algebraical function, and $a, b, \ldots, k$ commensurable numbers, can always be reduced to that of an algebraical function. In particular the integral

$$\int R(e^{ax}, e^{bx}, \ldots, e^{kx})\, dx,$$

* See, *e.g.*, Goursat, *Cours d'analyse*, ed. 2, vol. 1, pp. 257 *et seq.*

where R is rational, is always an elementary function. In the first place a substitution of the type $x = \alpha y$ will reduce it to the form

$$\int R(e^y)\,dy,$$

and then the substitution $e^y = z$ will reduce this integral to the integral of a rational function.

In particular, since $\cosh x$ and $\sinh x$ are rational functions of e^x, and $\cos x$ and $\sin x$ are rational functions of e^{ix}, the integrals

$$\int R(\cosh x, \sinh x)\,dx, \quad \int R(\cos x, \sin x)\,dx$$

are always elementary functions. In the second place the substitution just indicated is imaginary, and it is generally more convenient to use the substitution

$$\tan \tfrac{1}{2}x = t,$$

which reduces the integral to that of a rational function, since

$$\cos x = \frac{1-t^2}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}, \quad dx = \frac{2dt}{1+t^2}.$$

(ii) The integrals

$$\int R(\cosh x, \sinh x, \cosh 2x, \ldots\ldots \sinh mx)\,dx,$$

$$\int R(\cos x, \sin x, \cos 2x, \ldots\ldots \sin mx)\,dx,$$

are included in the two standard integrals above.

Let us consider some further developments concerning the integral

$$\int R(\cos x, \sin x)\,dx.^*$$

If we make the substitution $z = e^{ix}$, the subject of integration becomes a rational function $H(z)$, which we may suppose split up into

(*a*) a constant and certain positive and negative powers of z,

(*b*) groups of terms of the type

$$\frac{A_0}{z-a} + \frac{A_1}{(z-a)^2} + \ldots + \frac{A_n}{(z-a)^{n+1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The terms (i), when expressed in terms of x, give rise to a term

$$\Sigma(c_k \cos kx + d_k \sin kx).$$

In the group (1) we put $z = e^{ix}$, $a = e^{i\alpha}$ and, using the equation

$$\frac{1}{z-a} = \tfrac{1}{2}e^{-i\alpha}\{-1 - i\cot\tfrac{1}{2}(x-\alpha)\},$$

* See Hermite, *Cours d'analyse*, pp. 320 *et seq.*

we obtain a polynomial of degree $n+1$ in $\cot\frac{1}{2}(x-a)$. Since

$$\cot^2 x = -1 - \frac{d\cot x}{dx}, \qquad \cot^3 x = -\cot x - \frac{1}{2}\frac{d}{dx}(\cot^2 x), \ldots,$$

this polynomial may be transformed into the form

$$C + C_0 \cot\tfrac{1}{2}(x-a) + C_1 \frac{d}{dx}\cot\tfrac{1}{2}(x-a) + \ldots + C_n \frac{d^n}{dx^n}\cot\tfrac{1}{2}(x-a).$$

The function $R(\cos x, \sin x)$ is now expressed as a sum of a number of terms each of which is immediately integrable. The integral is a rational function of $\cos x$ and $\sin x$ if all the constants C_0 vanish; otherwise it includes a number of terms of the type

$$2C_0 \log \sin\tfrac{1}{2}(x-a).$$

Let us suppose for simplicity that $H(z)$, when split up into partial fractions, contains no terms of the types

$$C, \quad z^m, \quad z^{-m}, \quad (z-a)^{-p} \qquad (p>1).$$

Then

$$R(\cos x, \sin x) = C_0 \cot\tfrac{1}{2}(x-a) + D_0 \cot\tfrac{1}{2}(x-\beta) + \ldots,$$

and the constants $C_0, D_0, \ldots$ may be determined by multiplying each side of the equation by $\sin\frac{1}{2}(x-a)$, $\sin\frac{1}{2}(x-\beta), \ldots$ and making x tend to $a, \beta, \ldots$.

It is often convenient to use the equation

$$\cot\tfrac{1}{2}(x-a) = \cot(x-a) + \operatorname{cosec}(x-a)$$

which enables us to decompose the function R into two parts $U(x)$ and $V(x)$ such that

$$U(x+\pi) = U(x), \quad V(x+\pi) = -V(x).$$

If R has the period π, then V must vanish identically; if it changes sign when x is increased by π, then U must vanish identically. Thus we find without difficulty that, if $m<n$,

$$\frac{\sin mx}{\sin nx} = \frac{1}{2n}\sum_0^{2n-1} \frac{(-1)^k \sin ma}{\sin(x-a)} = \frac{1}{n}\sum_0^{n-1} \frac{(-1)^k \sin ma}{\sin(x-a)},$$

or

$$\frac{\sin mx}{\sin nx} = \frac{1}{n}\sum_0^{n-1} (-1)^k \sin ma \cot(x-a),$$

where $a = k\pi/n$, according as $m+n$ is odd or even.

Similarly

$$\frac{1}{\sin(x-a)\sin(x-b)\sin(x-c)} = \Sigma \frac{1}{\sin(a-b)\sin(a-c)\sin(x-a)},$$

$$\frac{\sin(x-d)}{\sin(x-a)\sin(x-b)\sin(x-c)} = \Sigma \frac{\sin(a-d)}{\sin(a-b)\sin(a-c)} \cot(x-a).$$

(iii) One of the most important integrals in applications is

$$\int \frac{dx}{a+b\cos x},$$

where a and b are real. This integral may be evaluated in the manner explained above, or by the transformation $\tan\frac{1}{2}x = t$. A more elegant method is the following. If $|a| > |b|$, we suppose a positive, and use the transformation

$$(a+b\cos x)(a-b\cos y)=a^2-b^2,$$

which leads to

$$\frac{dx}{a+b\cos x}=\frac{dy}{\sqrt{(a^2-b^2)}}.$$

If $|a| < |b|$, we suppose b positive, and use the transformation

$$(b\cos x+a)(b\cosh y-a)=b^2-a^2.$$

The integral

$$\int\frac{dx}{a+b\cos x+c\sin x}$$

may be reduced to this form by the substitution $x+\alpha=y$, where $\cot\alpha=b/c$. The forms of the integrals

$$\int\frac{dx}{(a+b\cos x)^n},\qquad \int\frac{dx}{(a+b\cos x+c\sin x)^n}$$

may be deduced by the use of formulae of reduction, or by differentiation with respect to a. The integral

$$\int\frac{dx}{(A\cos^2 x+2B\cos x\sin x+C\sin^2 x)^n}$$

is really of the same type, since

$$A\cos^2 x+2B\cos x\sin x+C\sin^2 x=\tfrac{1}{2}(A+C)+\tfrac{1}{2}(A-C)\cos 2x+B\sin 2x.$$

And similar methods may be applied to the corresponding integrals which contain hyperbolic functions, so that this type includes a large variety of integrals of common occurrence.

(iv) The same substitutions may of course be used when the subject of integration is an irrational function of $\cos x$ and $\sin x$, though sometimes it is better to use the substitutions $\cos x = t$, $\sin x = t$, or $\tan x = t$. Thus the integral

$$\int R(\cos x, \sin x, \sqrt{X})\,dx,$$

where

$$X=(a, b, c, f, g, h \between \cos x, \sin x, 1)^2,$$

is reduced to an elliptic integral by the substitution $\tan\tfrac{1}{2}x=t$. The most important integrals of this type are

$$\int\frac{R(\cos x, \sin x)\,dx}{\sqrt{(1-k^2\sin^2 x)}},\qquad \int\frac{R(\cos x, \sin x)\,dx}{\sqrt{(\alpha+\beta\cos x+\gamma\sin x)}}.$$

3. The integral

$$\int P(x, e^{ax}, e^{bx}, \dots, e^{kx})\,dx,$$

where $a, b, \dots, k$ are any numbers (commensurable or not), and P is a polynomial, is always an elementary function. For it is obvious that the integral can be reduced to the sum of a finite number of integrals of the type

$$\int x^p e^{Ax}\, dx;$$

and $$\int x^p e^{Ax}\, dx = \left(\frac{\partial}{\partial A}\right)^p \int e^{Ax}\, dx = \left(\frac{\partial}{\partial A}\right)^p \frac{e^{Ax}}{A}.$$

This type of integral includes a large variety of integrals, such as

$$\int x^m (\cos px)^\mu (\sin qx)^\nu\, dx, \quad \int x^m (\cosh px)^\mu (\sinh qx)^\nu\, dx,$$

$$\int x^m e^{-ax} (\cos px)^\mu\, dx, \quad \int x^m e^{-ax} (\sin qx)^\nu\, dx,$$

(m, μ, ν, being positive integers) for which formulae of reduction are given in text-books on the integral calculus.

Such integrals as

$$\int P(x, \log x)\, dx, \quad \int P(x, \text{arc} \sin x)\, dx, \ldots,$$

where P is a polynomial, may be reduced to particular cases of the above general integral by the obvious substitutions

$$x = e^y, \quad x = \sin y, \ldots.$$

4. Except for the two classes of functions considered in the three preceding paragraphs, there are no really general classes of transcendental functions which we can *always* integrate in finite terms, although of course there are innumerable particular forms which may be integrated by particular devices. There are however many classes of such integrals for which a systematic reduction theory may be given, analogous to the reduction theory for elliptic integrals. Such a reduction theory endeavours in each case

(i) to split up any integral of the class under consideration into the sum of a number of parts of which some are elementary and the others not;

(ii) to reduce the number of the latter terms to the least possible;

(iii) to prove that these terms are incapable of further reduction, and are genuinely new and independent transcendents.

As an example of this process we shall consider the integral

$$\int e^x R(x)\, dx$$

where $R(x)$ is a rational function of x.* The theory of partial

* See Hermite, *Cours d'analyse*, pp. 352 *et seq.*

fractions enables us to decompose this integral into the sum of a number of terms

$$A\int\frac{e^x}{x-a}\,dx,\quad A_m\int\frac{e^x}{(x-a)^{m+1}}\,dx,\ \dots,\quad B\int\frac{e^x}{x-b}\,dx,\ \dots.$$

Since

$$\int\frac{e^x}{(x-a)^{m+1}}\,dx=-\frac{e^x}{m\,(x-a)^m}+\frac{1}{m}\int\frac{e^x}{(x-a)^m}\,dx,$$

the integral may be further reduced so as to contain only

(i) a term $e^x S(x)$

where $S(x)$ is a rational function;

(ii) a number of terms of the type

$$\alpha\int\frac{e^x\,dx}{x-a}.$$

If all the constants α vanish, then the integral can be calculated in the finite form $e^x S(x)$. If they do not we can at any rate assert that the integral cannot be calculated *in this form**. For no such relation as

$$\alpha\int\frac{e^x dx}{x-a}+\beta\int\frac{e^x dx}{x-b}+\dots+\kappa\int\frac{e^x dx}{x-k}=e^x\,T(x),$$

where T is rational, can hold for all values of x. To see this it is only necessary to put $x=a+h$ and to expand in ascending powers of h. Then

$$\alpha\int\frac{e^x dx}{x-a}=\alpha e^a\int\frac{e^h}{h}\,dh$$

$$=\alpha e^a(\log h+h+\dots),$$

and no *logarithm* can occur in any of the other terms†.

Consider, for example, the integral

$$\int e^x\left(1-\frac{1}{x}\right)^3 dx.$$

This is equal to $$e^x-3\int\frac{e^x}{x}\,dx+3\int\frac{e^x}{x^2}\,dx-\int\frac{e^x}{x^3}\,dx,$$

and since $$3\int\frac{e^x}{x^2}\,dx=-\frac{3e^x}{x}+3\int\frac{e^x}{x}\,dx,$$

and

$$-\int\frac{e^x}{x^3}\,dx=\frac{e^x}{2x^2}-\frac{1}{2}\int\frac{e^x}{x^2}\,dx=\frac{e^x}{2x^2}+\frac{e^x}{2x}-\frac{1}{2}\int\frac{e^x}{x}\,dx,$$

* See the remarks at the end of this paragraph.

† It is not difficult to give a purely algebraical proof on the lines of IV., § 2.

we obtain finally

$$\int e^x\left(1-\frac{1}{x}\right)^3 dx=e^x\left(1-\frac{7}{2x}+\frac{1}{2x^2}\right)-\frac{1}{2}\int\frac{e^x}{x}\,dx.$$

Similarly it will be found that

$$\int e^x\left(1-\frac{2}{x}\right)^2 dx=2e^x\left(\frac{1}{2}-\frac{2}{x}\right),$$

this integral being an elementary function.

Since $$\int\frac{e^x}{x-a}\,dx=e^a\int\frac{e^y}{y}\,dy,$$

if $x=y+a$, all integrals of this kind may be made to depend on known functions and on the single transcendent

$$\int\frac{e^x}{x}\,dx,$$

which is usually denoted by $Li\,e^x$ and is of great importance in the theory of numbers. The question of course arises as to whether this integral is not itself an elementary function.

Now Liouville* has proved the following theorem: '*if y is any algebraical function of x, and*

$$\int e^x y\,dx$$

is an elementary function, then

$$\int e^x y\,dx=e^x(\alpha+\beta y+\ldots+\lambda y^{n-1}),$$

α, β, ..., λ being rational functions of x and n the degree of the algebraical equation which determines y as a function of x'.

Liouville's proof rests on the same general principles as do those of the corresponding theorems concerning the integral $\int y\,dx$. It will be observed that no logarithmic terms can occur, and that the theorem is therefore very similar to that which holds for $\int y\,dx$ in the simple case in which the integral is *algebraical*. The argument which shows that no logarithmic terms occur is substantially the same as that which shows that, when they occur in the integral of an algebraical function, they must occur linearly. In this case the occurrence of the exponential factor precludes even this possibility, since differentiation will not eliminate logarithms when they occur in the form

$$e^x\log f(x).$$

* 'Mémoire sur l'intégration d'une classe de fonctions transcendantes', *Journal für Mathematik*, vol. 13, 1835, pp. 93–118. Liouville shows how the integral, when of this form, may always be calculated by elementary methods.

In particular, if y is a rational function, then the integral must be of the form

$$e^x R(x)$$

and this we have already seen to be impossible. Hence the 'logarithm-integral'

$$Li\, e^x = \int \frac{e^x}{x}\, dx = \int^{e^x} \frac{dy}{\log y}$$

is really a new transcendent, which cannot be expressed in finite terms by means of elementary functions; and the same is true of all integrals of the type

$$\int e^x R(x)\, dx$$

which cannot be calculated in finite terms by means of the process of reduction sketched above.

The integrals

$$\int \sin x\, R(x)\, dx, \quad \int \cos x\, R(x)\, dx$$

may be treated in a similar manner. Either the integral is of the form

$$\cos x\, R_1(x) + \sin x\, R_2(x)$$

or it consists of a term of this kind together with a number of terms which involve the transcendents

$$\int \frac{\cos x}{x}\, dx, \quad \int \frac{\sin x}{x}\, dx,$$

which are called the cosine-integral and sine-integral of x, and denoted by $Ci\, x$ and $Si\, x$. These transcendents are of course not fundamentally distinct from the logarithm-integral.

5. Liouville has gone further and shown that it is always possible to determine whether the integral

$$\int (Pe^p + Qe^q + \dots + Te^t)\, dx,$$

where $P, Q, \dots, T, p, q, \dots, t$ are algebraical functions, is an elementary function, and to obtain the integral in case it is one*. The most general theorem which has been proved in this region of mathematics, and which is also due to Liouville, is the following.

* An interesting particular result is that the 'error function' $\int e^{-x^2} dx$ is not an elementary function.

'If y, z, ... are functions of x whose differential coefficients are algebraical functions of x, y, z, ..., and F denotes an algebraical function, and if

$$\int F(x, y, z, \ldots)\,dx$$

is an elementary function, then it is of the form

$$t + A \log u + B \log v + \ldots,$$

where t, u, v, ... are algebraical functions of x, y, z, ... If the differential coefficients are rational in x, y, z, ..., and F is rational, then t, u, v, ... are rational in x, y, z, ...'

Thus for example the theorem applies to

$$F(x, e^x, e^{e^x}, \log x, \log\log x, \cos x, \sin x),$$

since, if the various arguments of F are denoted by x, y, z, ξ, η, ζ, θ, we have

$$\frac{dy}{dx} = y, \qquad \frac{dz}{dx} = yz, \qquad \frac{d\xi}{dx} = \frac{1}{x},$$

$$\frac{d\eta}{dx} = \frac{1}{x\xi}, \qquad \frac{d\zeta}{dx} = -\sqrt{(1-\zeta^2)}, \qquad \frac{d\theta}{dx} = \sqrt{(1-\theta^2)}.$$

The proof of the theorem does not involve ideas different in principle from those which have been employed continually throughout the preceding pages.

6. As a final example of the manner in which these ideas may be applied, we shall consider the following question:

'in what circumstances is

$$\int R(x) \log x\,dx,$$

where R is rational, an elementary function?'

In the first place the integral must be of the form

$$R_0(x, \log x) + A_1 \log R_1(x, \log x) + A_2 \log R_2(x, \log x) + \ldots.$$

A general consideration of the form of the differential coefficient of this expression, in which $\log x$ must only occur linearly and multiplied by a rational function, leads us to anticipate that (i) $R_0(x, \log x)$ must be of the form

$$S(x)(\log x)^2 + T(x) \log x + U(x),$$

where S, T, and U are rational, and (ii) R_1, R_2, ... must be rational functions of x only; so that the integral can be expressed in the form

$$S(x)(\log x)^2 + T(x) \log x + U(x) + \Sigma B_k \log (x - a_k).$$

Differentiating, and comparing the result with the subject of integration, we obtain the equations

$$S'=0,\quad \frac{2S}{x}+T'=R,\quad \frac{T}{x}+U'+\Sigma\frac{B_k}{x-a_k}=0.$$

Hence S is a constant, say $\frac{1}{2}C$, and

$$T=\int\left(R-\frac{C}{x}\right)dx.$$

We can always determine by means of elementary operations, as in IV., § 4, whether this integral is rational for any value of C or not. If not, then the given integral is not an elementary function. If T is rational, then we must calculate its value, and substitute it in the integral

$$U=-\int\left\{\frac{T}{x}+\Sigma\frac{B_k}{x-a_k}\right\}dx=-\int\frac{T}{x}dx-\Sigma B_k\log(x-a_k),$$

which must be rational for some value of the arbitrary constant implied in T. We can calculate the rational part of

$$\int\frac{T}{x}dx:$$

the transcendental part must be cancelled by the logarithmic terms

$$\Sigma B_k\log(x-a_k).$$

The necessary and sufficient condition that the original integral should be an elementary function is therefore that R should be of the form

$$\frac{C}{x}+\frac{d}{dx}\{R_1(x)\},$$

where C is a constant and R_1 is rational. That the integral is in this case such a function becomes obvious if we integrate by parts, for

$$\int\left(\frac{C}{x}+R_1'\right)\log x\,dx=\tfrac{1}{2}C(\log x)^2+R_1\log x-\int\frac{R_1}{x}dx.$$

In particular

$$\text{(i)}\quad\int\frac{\log x}{x-a}dx,\qquad\qquad \text{(ii)}\quad\int\frac{\log x}{(x-a)(x-b)}dx,$$

are not elementary functions unless in (i) $a=0$ and in (ii) $b=a$. If the integral is elementary then the integration can always be carried out, with the same reservation as was necessary in the case of rational functions.

It is evident that the problem considered in this paragraph is but one of a whole class of similar problems. The reader will find it instructive to formulate and consider such problems for himself.

7. It will be obvious by now that the number of classes of transcendental functions whose integrals are always elementary is very small, and that such integrals as

$$\int f(x, e^x)\,dx, \qquad \int f(x, \log x)\,dx,$$

$$\int f(x, \cos x, \sin x)\,dx, \quad \int f(e^x, \cos x, \sin x)\,dx,$$

$$\cdots\cdots\cdots\cdots,$$

where f is algebraical, or even rational, are generally new transcendents. These new transcendents, like the transcendents (such as the elliptic integrals) which arise from the integration of algebraical functions, are in many cases of great interest and importance. They may often be expressed by means of infinite series or definite integrals, or their properties may be studied by means of the integral expressions which define them. The very fact that such a function is *not* an elementary function in so far enhances its importance. And when such functions have been introduced into analysis new problems of integration arise in connection with them. We may enquire, for example, under what circumstances an elliptic integral or elliptic function, or a combination of such functions with elementary functions, can be integrated in finite terms by means of elementary and elliptic functions. But before we can be in a position to restate the fundamental problem of the Integral Calculus in any such more general form, it is essential that we should have disposed of the particular problem formulated in Section III.

APPENDIX I

BIBLIOGRAPHY

The following is a list of the memoirs by Abel, Liouville and Tschebyschef which have reference to the subject matter of this tract.

N. H. Abel

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2. 'Précis d'une théorie des fonctions elliptiques', *Journal für Mathematik*, vol. 4, 1829, pp. 236-277, 309–348 (*Œuvres*, vol. 1, pp. 518–617).
3. 'Théorie des transcendantes elliptiques', *Œuvres*, vol. 2, pp. 87–188.

J. Liouville

1. 'Mémoire sur la classification des transcendantes, et sur l'impossibilité d'exprimer les racines de certaines équations en fonction finie explicite des coefficients', *Journal de mathématiques*, ser. 1, vol. 2, 1837, pp. 56–104.
2. 'Nouvelles recherches sur la détermination des intégrales dont la valeur est algébrique', *ibid.*, vol. 3, 1838, pp. 20–24 (previously published in the *Comptes Rendus*, 28 Aug. 1837).
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4. 'Sur l'intégration des différentielles irrationnelles', *ibid.*, pp. 242–246 (*Œuvres*, vol. 1, pp. 511–514; previously published in the *Comptes Rendus*, 9 July 1860).

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K. Weierstrass

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G. Zolotareff

'Sur la méthode d'intégration de M. Tschebyschef', *Journal de mathématiques*, ser. 2, vol. 19, 1874, pp. 161–188.

Further information concerning pseudo-elliptic integrals, and degenerate cases of Abelian integrals generally, will be found in a number of short notes by Dolbnia, Kapteyn and Ptaszycki in the *Bulletin des sciences mathématiques*, and by Goursat, Gunther, Picard, Poincaré, and Raffy in the *Bulletin de la Société Mathématique de France*, in Legendre's *Traité des fonctions elliptiques* (vol. 1, ch. 26), in Halphen's *Traité des fonctions elliptiques* (vol. 2, ch. 14), and in Enneper's *Elliptische Funktionen.* The literature concerning the general theory of algebraical functions and their integrals is too extensive to be summarised here: the reader may be referred to Appell and Goursat's *Théorie des fonctions algébriques*, and Wirtinger's article *Algebraische Funktionen und ihre Integrale* in the *Encyclopädie der Mathematischen Wissenschaften*, II B 2.

APPENDIX II

ON ABEL'S PROOF OF THE THEOREM OF V., § 11

Abel's proof (*Œuvres*, vol. 1, p. 545) is as follows*:

We have

$$\psi(x, u) = 0 \quad \text{.................................(1)},$$

where ψ is an irreducible polynomial of degree m in u. If we make use of the equation $f(x, y) = 0$, we can introduce y into this equation, and write it in the form

$$\phi(x, y, u) = 0 \quad \text{..............................(2)},$$

where ϕ is a polynomial in the three variables x, y, and u†; and we can suppose ϕ, like ψ, of degree m in u and irreducible, that is to say not divisible by any polynomial of the same form which is not a constant multiple of ϕ or itself a constant.

From $f=0$, $\phi=0$ we deduce

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0, \qquad \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} + \frac{\partial \phi}{\partial u}\frac{du}{dx} = 0;$$

and, eliminating $\frac{dy}{dx}$, we obtain an equation of the form

$$\frac{du}{dx} = \frac{\lambda(x, y, u)}{\mu(x, y, u)},$$

where λ and μ are polynomials in x, y, and u. And in order that u should be an integral of y it is necessary and sufficient that

$$\lambda - y\mu = 0 \quad \text{.................................(3)}.$$

Abel now applies Lemma (2) of § 11, or rather its analogue for polynomials in u whose coefficients are polynomials in x and y, to the two polynomials ϕ and $\lambda - y\mu$, and infers that *all* the roots $u, u', \ldots$ of $\phi=0$ satisfy (3). From this he deduces that $u, u', \ldots$ are all integrals of y, and so that

$$\frac{u + u' + \ldots}{m+1} \quad \text{.................................(4)}$$

* The theorem with which Abel is engaged is a very much more general theorem.

† 'Or, au lieu de supposer ces coefficiens rationnels en x, nous les supposerons rationnels en x, y; *car cette supposition permise simplifiera beaucoup le raisonnement*'.

is an integral of y. As (4) is a symmetric function of the roots of (2), it is a rational function of x and y, whence his conclusion follows*.

It will be observed that the hypothesis that (2) does actually involve y is essential, if we are to avoid the absurd conclusion that u is necessarily *a rational function of x only*. On the other hand it is not obvious how the presence of y in ϕ affects the other steps in the argument.

The crucial inference is that which asserts that because the equations $\phi=0$ and $\lambda-y\mu=0$, considered as equations in u, have a root in common, and ϕ is irreducible, therefore $\lambda-y\mu$ is divisible by ϕ. This inference is invalid.

We could only apply the lemma in this way if the equation (3) were satisfied by one of the roots of (2) *identically*, that is to say for all values of x and y. But this is not the case. The equations are satisfied by the same value of u *only when x and y are connected by the equation* (1).

Suppose, for example, that

$$y=\frac{1}{\sqrt{(1+x)}}, \quad u=2\sqrt{(1+x)}.$$

Then we may take

$$f=(1+x)y^2-1,$$

$$\psi=u^2-4(1+x),$$

and

$$\phi=uy-2.$$

Differentiating the equations $f=0$ and $\phi=0$, and eliminating $\frac{dy}{dx}$, we find

$$\frac{du}{dx}=\frac{u}{2(1+x)}=\frac{\lambda}{\mu}.$$

Thus

$$\phi=uy-2, \quad \lambda-y\mu=u-2y(1+x);$$

and these polynomials have a common factor only in virtue of the equation $f=0$.

* Bertrand (*Calcul intégral*, ch. 5) replaces the last step in Abel's argument by the observation that if u and u' are both integrals of y then $u-u'$ is constant (cf. p. 39, bottom). It follows that the degree of the equation which defines u can be decreased, which contradicts the hypothesis that it is irreducible.

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PREFACE TO THE SECOND EDITION

THE present edition of this tract embodies a large number of alterations and additions. In particular I have rewritten Section VI completely, and hope it may now be useful as an introduction, from a special point of view, to a large field of modern research.

I should like to add a few words concerning the motives of Sections III—V, which form the most characteristic part of the tract, and I can make my point best by reference to a particular problem. Suppose that the problem is that of determining the behaviour of the power series $\Sigma \phi(n) x^n$ when x tends to unity. It is usual to delimit the problem in one or other of two ways. One is to restrict $\phi(n)$ by 'conditions of inequality', to suppose, for example, that $\phi(n)$ and a certain number of its derivatives or differences are monotonic functions of specified signs. The other is to confine our attention to special forms of $\phi(n)$, such as $n^\alpha (\log n)^\beta (\log\log n)^\gamma \ldots$, sufficiently general to illustrate the principal questions at issue.

There is, however, a third point of view which is often advantageous in the discussion of problems of this character. We may suppose that $\phi(n)$ is any function of some standard corpus whose rate of increase is not too large; and the natural corpus to select is the corpus of 'L-functions', that is to say of functions finitely definable by logarithms and exponentials. Thus, in the particular problem which I have mentioned, we may suppose $\phi(n)$ to be any L-function whose increase does not exceed that of all powers of n. In this way we may hope to prove theorems, not of course exhaustive, but including all the standard examples as particular cases. This point of view is adopted by implication in much of du Bois-Reymond's work, and it is that which is usually adopted here. It is, however, obviously necessary to begin by an exact and general investigation of the properties of L-functions, and this du Bois-Reymond omitted. The first essential theorem, for example, is that which appears here as Theorem 13. This theorem may be verified immediately in any particular case, but du Bois-Reymond never proves it and, so far as I know, no general proof had been given before the publication of this tract.

I am much indebted to Mr E. C. Titchmarsh and Mr A. Oppenheim for suggestions made in the course of correction of the proofs.

G. H. H.

20 *February*, 1924.

CONTENTS

I

INTRODUCTION

1.1. THE notions of the 'order of greatness' or 'order of smallness' of a function $f(n)$ of a positive integral variable n, when n is 'large', or of a function $f(x)$ of a continuous variable x, when x is 'large' or 'small' or 'nearly equal to a', are important even in the most elementary stages of mathematical analysis*. We learn there that x^2 tends to infinity with x, and moreover that x^2 tends to infinity *more rapidly than* x, *i.e.* that the ratio x^2/x tends to infinity also; and that x^3 tends to infinity more rapidly than x^2, and so on indefinitely. We are thus led to the idea of a 'scale of infinity' (x^n) formed by the functions $x, x^2, x^3, \ldots, x^n, \ldots$. This scale may be supplemented and to some extent completed by the interpolation of non-integral powers of x. But there are functions whose rates of increase cannot be measured by any of the functions of our scale, even when thus completed. Thus $\log x$ tends to infinity more slowly, and e^x more rapidly, than *any* power of x; and $x/(\log x)$ tends to infinity more slowly than x, but more rapidly than any power of x less than the first.

As we proceed further in analysis, and come into contact with its modern developments, such as the theory of Fourier's series, the theory of integral functions, or the theory of singular points of analytic functions in general, the importance of these ideas becomes greater and greater. It is the systematic study of them, the investigation of general theorems concerning them and ready methods of handling them, that is the subject of Paul du Bois-Reymond's *Infinitärcalcül* or 'calculus of infinities'.

1.2. Let us suppose that f and ϕ are two functions of the continuous variable x, defined for all values of x from a certain value x_0 onwards. Further, let us suppose that f and ϕ are positive, continuous, and steadily increasing, and tend to infinity with x; and let us consider the behaviour of the ratio f/ϕ when $x \to \infty$. We can distinguish four cases.

(i) If $f/\phi \to \infty$, we shall say that the *order*, or the *rate of increase*, or simply the *increase*, of f is greater than that of ϕ, and write

$$f \succ \phi.$$

(ii) If $f/\phi \to 0$, we shall say that the increase of f is less than that of ϕ, and write

$$f \prec \phi.$$

* See, for instance, Hardy, 1, 360.

(iii) If f/ϕ remains, for all values of x from a certain value x_1 onwards*, between two positive numbers δ and Δ, so that $0 < \delta < f/\phi < \Delta$, we shall say that the increase of f is equal to that of ϕ, and write

$$f \asymp \phi.$$

It may happen, in this case, that f/ϕ tends to a definite limit. If this is so, we shall write $$f \succcurlyeq\!\!\preccurlyeq \phi.$$

Finally, if this limit is *unity*, we shall write

$$f \sim \phi.$$

When we can compare the increase of f with that of some standard function ϕ by means of a relation of the type $f \asymp \phi$, we shall say that ϕ *measures*, or simply *is*, the increase of f. Thus we shall say that the increase of $2x^2 + x + 3$ is x^2.

It often happens that f/ϕ is monotonic (*i.e.* steadily increasing or steadily decreasing) as well as f and ϕ themselves. In this case f/ϕ must tend to infinity, or to zero, or to a positive limit: so that $f \succ \phi$ or $f \prec \phi$ or $f \succcurlyeq\!\!\preccurlyeq \phi$. We shall see in a moment that this is not true in general.

(iv) It may happen that f/ϕ neither tends to infinity nor to zero, nor remains between positive bounds.

Suppose, for example, that ϕ_1, ϕ_2 are two continuous and increasing functions such that $\phi_1 \succ \phi_2$. A glance at the figure (Fig. 1) will probably show with sufficient clearness how we can construct, by means of a 'staircase' of straight or curved lines, running backwards and forwards between the graphs of ϕ_1 and ϕ_2, the graph of a steadily increasing function f such that $f = \phi_1$ for $x = x_1, x_3, \ldots$ and $f = \phi_2$ for $x = x_2, x_4, \ldots$. Then $f/\phi_1 = 1$ for

$$x = x_1, x_3, \ldots,$$

but assumes for $x = x_2, x_4, \ldots$ values which decrease beyond all limit; while $f/\phi_2 = 1$ for $x = x_2, x_4, \ldots$, but assumes for $x = x_1, x_3, \ldots$ values which increase beyond all limit; and f/ϕ, where ϕ is a function, such as $\sqrt{(\phi_1\phi_2)}$, for which $\phi_1 \succ \phi \succ \phi_2$, assumes both values which increase beyond all limit and values which decrease beyond all limit.

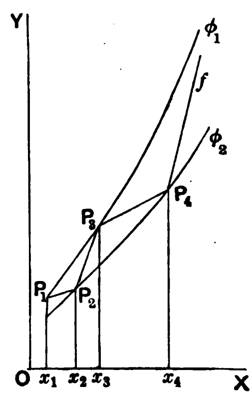

Fig. 1

* No mention of x_1 is really necessary when (as is supposed in the text) f and ϕ are positive and continuous. There are then numbers δ_1 and Δ_1 such that $0 < \delta_1 < f/\phi < \Delta_1$ for $x_0 \leqq x < x_1$, and $0 < \delta < f/\phi < \Delta$ for $x \geqq x_1$ implies $0 < \delta_2 < f/\phi < \Delta_2$, where $\delta_2 = \text{Min}(\delta, \delta_1)$, $\Delta_2 = \text{Max}(\Delta, \Delta_1)$, for $x \geqq x_0$.

It is however often convenient to extend our definitions to more general cases in which this argument would be invalid, and we retain the unnecessary words in order that the definitions may be more immediately adaptable.

Later on (§ 4.43) we shall meet with cases of this kind in which the functions are defined by explicit analytical formulae.

1.3. If a positive constant δ can be found such that $f > \delta\phi$ for all sufficiently large values of x, we shall write

$$f \succcurlyeq \phi;$$

and if a positive constant Δ can be found such that $f < \Delta\phi$ for all sufficiently large values of x, we shall write

$$f \preccurlyeq \phi.$$

If $f \succcurlyeq \phi$ and $f \preccurlyeq \phi$, then $f \asymp \phi$.

It is however important to observe that $f \succcurlyeq \phi$ is not logically equivalent to the negation of $f \prec \phi$. The relations $f \succcurlyeq \phi$, $f \prec \phi$ are mutually exclusive, but not exhaustive; the first implies the negation of the second, but the converse is not true. Again, $f \succcurlyeq \phi$ is not equivalent to the alternative '$f \succ \phi$ *or* $f \asymp \phi$'. Each of these points may be illustrated by the example at the end of § 1.2. Here $f \succcurlyeq \phi_1$ and $f \prec \phi_1$ are both false; and $f \succcurlyeq \phi_2$, but neither $f \succ \phi_2$ nor $f \asymp \phi_2$ is true. In the language of upper and lower limits, $f \succcurlyeq \phi$ means

$$\lambda = \underline{\lim}\,\frac{f}{\phi} > 0$$

and '$f \prec \phi$ is false' means

$$\Lambda = \overline{\lim}\,\frac{f}{\phi} > 0;$$

while to assert '$f \succ \phi$ or $f \asymp \phi$' is to assert that $\lambda > 0$ and that, if λ is finite, Λ is also finite.

The reader will have no difficulty in proving the following theorems. There are many other simple theorems of the same character, but these seem the most important.

(*a*) *If* $f \succ \phi$, $\phi \succcurlyeq \psi$, *then* $f \succ \psi$.
(*b*) *If* $f \succcurlyeq \phi$, $\phi \succ \psi$, *then* $f \succ \psi$.
(*c*) *If* $f \succcurlyeq \phi$, $\phi \succcurlyeq \psi$, *then* $f \succcurlyeq \psi$.
(*d*) *If* $f \asymp \phi$, $\phi \asymp \psi$, *then* $f \asymp \psi$.
(*e*) *If* $f \succcurlyeq \phi$, *then* $f + \phi \asymp f$.
(*f*) *If* $f \succ \phi$, *then* $f - \phi \asymp f$.
(*g*) *If* $f \succ \phi$, $f_1 \succ \phi_1$, *then* $f + f_1 \succ \phi + \phi_1$.
(*h*) *If* $f \succ \phi$, $f_1 \asymp \phi_1$, *then* $f + f_1 \succcurlyeq \phi + \phi_1$.
(*i*) *If* $f \asymp \phi$, $f_1 \asymp \phi_1$, *then* $f + f_1 \asymp \phi + \phi_1$.
(*j*) *If* $f \succ \phi$, $f_1 \succcurlyeq \phi_1$, *then* $ff_1 \succ \phi\phi_1$.
(*k*) *If* $f \asymp \phi$, $f_1 \asymp \phi_1$, *then* $ff_1 \asymp \phi\phi_1$.

He will also find it instructive to state for himself a series of similar theorems involving also the symbols $\mathrel{\overset{\frown}{\underset{\smile}{=}}}$ and $\sim$.

1.4. So far we have supposed that the functions considered all tend to infinity with x. There is nothing to prevent us from including cases in which f or ϕ tends steadily to zero, or to a limit other than zero; thus we may write $x \succ 1$, or $x \succ 1/x$, or $1/x \succ 1/x^2$. Bearing this in mind, the reader should frame a series of theorems similar to those of § 1.3 but involving quotients instead of sums or products.

It is also convenient to extend our definitions so as to apply to *negative* functions which tend steadily to $-\infty$, or to 0 or to some other limit. In such cases we make no distinction, when using the symbols $\succ$, $\prec$, $\asymp$, $\succcurlyeq\!\!\preccurlyeq$, between the function and its modulus: thus we write $-x \prec -x^2$ or $-1/x \prec 1$, meaning thereby exactly the same as by $x \prec x^2$ or $1/x \prec 1$. But $f \sim \phi$ is to be interpreted as a statement about the actual functions and not about their moduli.

It will be well now to lay down the principle that functions referred to in this tract, from this point onwards, are to be understood, unless the contrary is expressly stated or obviously implied, to be positive, continuous, and monotonic, increasing if they tend to infinity, and decreasing if they tend to zero. But it is sometimes convenient to depart from these conventions. We may abandon the restriction to continuous functions, writing, for example,

$$[x] \sim x, \quad \pi(x) \prec x,$$

where $[x]$ is the integral part of x and $\pi(x)$ the number of primes which do not exceed x. Or we may write

$$1 + \sin x \prec x, \; x^2 \succ x \sin x,$$

meaning by the first formula, for example, that $(1 + \sin x)/x \to 0$. We may even apply our notation to complex functions, writing $e^{ix} \prec x$ or $e^{ix} \asymp 1$. The reader will find no difficulty in modifying the definitions in the appropriate manner.

There are other possibilities to be considered. We have so far confined our attention to functions of a continuous variable x which tends to $+\infty$. This case may be held to include one which is perhaps even more important in applications, viz. that of functions of the positive integral variable n. We have only to disregard non-integral values of x. Thus $n! \succ n^2$, $-1/n \prec n$.

Finally, by putting $x = -y$, $x = 1/y$, or $x = 1/(y-a)$, we are led to consider functions of a continuous variable y which tends to $-\infty$ or 0 or a. The reader will easily supply the necessary modifications of detail.

In what follows we shall generally state and prove our theorems only for the case with which we started, that of continuous and increasing functions of a continuous variable which tends to infinity, and shall leave to the reader the task of formulating the corresponding theorems for the other cases.

1.5. There are some other symbols which we shall sometimes find it convenient to use in special senses. By

$$O(\phi)$$

we shall denote a function f, otherwise unspecified, but such that

$$|f| < K\phi,$$

where K is a constant and ϕ a positive function of x. This notation was first used by Bachmann*, though its general adoption is due to the influence of Landau. Thus

$$x+1 = O(x), \quad x = O(x^2), \quad \sin x = O(1).$$

It is clear that the three assertions

$$f = O(\phi), \quad |f| < K\phi, \quad f \preccurlyeq \phi$$

are equivalent to one another. By

$$o(\phi)$$

we shall, again following Landau †, denote a function f such that $f/\phi \to 0$. Thus

$$x = o(x^2), \quad 1 = o(x), \quad \sin x = o(x)$$

and

$$f = o(\phi), \quad f/\phi \to 0, \quad f \prec \phi$$

are equivalent.

We shall follow Borel ‡ in using the same letter K in a whole series of inequalities to denote a positive number, independent of the variable under consideration, but not necessarily the same in all inequalities where it occurs. Thus

$$\sin x < K, \quad 2x+1 < Kx, \quad x^m < Ke^x \quad (x \geqq 1).$$

If we use K thus in any finite number of inequalities which (like the first two above) do not involve any variables other than x, or whatever other variable we are considering, then all the values of K lie between two numbers K_1 and K_2: thus K_1 might be 10^{-10} and K_2 be 10^{10}. In this case all the K's satisfy $0 < K_1 < K < K_2$, and every relation $f < K\phi$ might be replaced by $f < K_2\phi$, and every relation $f > K\phi$ by $f > K_1\phi$. But we shall also have occasion to use K in equalities which (like the third above) involve a parameter (here m). In this case K, though independent of x, is a function of m. Suppose that a finite number of parameters $\alpha, \beta, \ldots$ occur in this way in this tract. Then if we give any

* Bachmann, **1**, 401. † Landau, **1**, 61. ‡ Borel, **6** and **2**, 105.

special system of values to α, β, ..., we can determine K_1, K_2 as above. Thus all our K's satisfy

$$0 < K_1(\alpha, \beta, \ldots) < K < K_2(\alpha, \beta, \ldots),$$

where K_1, K_2 are positive functions of α, β, ... defined for any permissible set of values of those parameters. But K_1 may have the lower bound zero, and K_2 may be unbounded. We can then, by choosing α, β, ... appropriately, make K_1 as small and K_2 as large as we please.

When a function f possesses a property for all values of x greater than some definite value, this value of course depending on the function and the property, we shall say that f possesses the property for $x > x_0$. Thus

$$x > 100 \quad (x > x_0), \qquad e^x > 100x^2 \quad (x > x_0).$$

We shall use δ and Δ to denote arbitrary but fixed positive numbers, using δ when we wish to emphasize the possible smallness of the number, and Δ when we wish to emphasize its possible largeness. Thus

$$f < \delta\phi \quad (x > x_0)$$

means 'however small δ, we can find x_0 so that $f < \delta\phi$ for $x > x_0$', *i.e.* means the same as $f \prec \phi$; and

$$(\log x)^\Delta \prec x^\delta$$

means 'any power of $\log x$ (however great) tends to infinity more slowly than any positive power of x (however small)'.

Finally, we denote by ϵ a function (of a variable or variables indicated by the context or by a suffix) whose limit is zero when the variable or variables are made to tend to infinity or to their limits in the way we happen to be considering. Thus ϵ means the same as $o(1)$, and

$$f = \phi(1+\epsilon), \; f \sim \phi, \; f = \phi + o(\phi)$$

are equivalent to one another.

1.6. In order to become familiar with the use of the symbols defined in the preceding sections the reader is advised to verify the following relations, in which $P_m(x)$, $Q_n(x)$ denote polynomials whose degrees are m and n and whose leading coefficients are positive:

$$P_m(x) \succ Q_n(x) \;\; (m > n), \quad P_m(x) \asymp Q_n(x) \;\; (m = n),$$

$$P_m(x) \asymp x^m, \quad P_m(x)/Q_n(x) \asymp x^{m-n},$$

$$\sqrt{(ax^2+2bx+c)} \asymp x \;\; (a>0), \quad \sqrt{(x+a)} \sim \sqrt{x}, \quad \sqrt{(x+a)} - \sqrt{x} \sim \tfrac{1}{2}ax^{-\frac{1}{2}},$$

$$e^x \succ x^\Delta, \quad e^{x^2} \succ e^{\Delta x}, \quad e^{e^x} \succ e^{x^\Delta}, \quad \log x \prec x^\delta, \quad \log\log x \prec (\log x)^\delta,$$

$$\log P_m(x) \asymp \log Q_n(x), \quad \log\log P_m(x) \sim \log\log Q_n(x),$$

$$x + a\sin x \sim x, \quad x(a+\sin x) \asymp x \;\; (a>1),$$

$$e^{a+\sin x} \asymp 1, \quad \cosh x \sim \sinh x \sim \tfrac{1}{2}e^x, \quad \cosh(x+a) \asymp \cosh x,$$

$$x^{\Delta} = o(e^{\delta x}), \quad (\log x)^{\Delta} = o\{e^{(\log x)^{\delta}}\}, \quad x^{\Delta} = o\{e^{(\log x)^{1+\delta}}\},$$

$$1+\frac{1}{2}+\dots+\frac{1}{n} \sim \log n, \quad 1+\frac{1}{2}+\dots+\frac{1}{n}-\log n \asymp 1,$$

$$n! \prec n^n, \quad n! \succ e^{\Delta n}, \quad n! = n^{n^{1+\epsilon}} = n^{n(1+\epsilon)},$$

$$n! \sim n^{n+\frac{1}{2}}e^{-n}\sqrt{(2\pi)}, \quad n!\,(e/n)^n = (1+\epsilon)\sqrt{(2\pi n)},$$

$$\int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}, \quad \int_2^x \frac{dt}{\log t} = \frac{x}{\log x} + O\left\{\frac{x}{(\log x)^2}\right\}, \quad \int_3^x \frac{dt}{\log\log t} \sim \frac{x}{\log\log x}.$$

II

SCALES OF INFINITY IN GENERAL

2.1. If we start from a function ϕ, such that $\phi \succ 1$, we can, in a variety of ways, form a series of functions

$$\phi_1 = \phi, \ \phi_2, \ \phi_3, \ \dots, \ \phi_n, \ \dots$$

such that the increase of each function is greater than that of its predecessor. Such a sequence of functions we shall denote for shortness by (ϕ_n).

One obvious method is to take $\phi_n = \phi^n$. Another is as follows: If $\phi \succ x$, it is clear that

$$\phi\{\phi(x)\}/\phi(x) \to \infty,$$

and so $\phi_2(x) = \phi\phi(x) \succ \phi(x)$; similarly $\phi_3(x) = \phi\phi_2(x) \succ \phi_2(x)$, and so on.

Thus the first method, with $\phi = x$, gives the scale $x, x^2, x^3, \dots$ or (x^n); the second, with $\phi = x^2$, gives the scale $x^2, x^4, x^8, \dots$ or (x^{2^n}). In this case the second scale is merely a selection from the terms of the first. With $\phi = e^x$, the two methods give the scales $e^x, e^{2x}, e^{3x}, \dots$ and

$$e^x, \ e^{e^x}, \ e^{e^{e^x}}, \ \dots.$$

Here the second term of the second scale is of greater increase than any term of the first.

These scales are *enumerable* scales, formed by a simple progression of functions. We can also, of course, by replacing the integral parameter n

by a continuous parameter a, define scales containing a non-enumerable multiplicity of functions: the simplest is (x^a), where a is any positive number. But such scales play a subordinate part in the theory.

It is obvious that we can always insert a new term (and therefore, of course, any number of new terms) in a scale at the beginning or between any two terms: thus $\sqrt{\phi}$ (or ϕ^a, where a is any positive number less than unity) has an increase less than that of any term of the scale, and $\sqrt{(\phi_n \phi_{n+1})}$ or $\phi_n^a \phi_{n+1}^{1-a}$ has an increase intermediate between those of ϕ_n and ϕ_{n+1}. A less obvious and more important theorem is the following.

Theorem 1*. *Given any ascending scale of increasing functions* ϕ_n, *i.e. a series of functions such that* $\phi_1 \prec \phi_2 \prec \phi_3 \prec \dots$, *we can always find a function* f *which increases more rapidly than any function of the scale, i.e. which satisfies the relation* $\phi_n \prec f$ *for all values of* n.

In view of the fundamental importance of this theorem we shall give two entirely different proofs.

2.21. We know that $\phi_{n+1} \succ \phi_n$ for all values of n, but this, of course, does not necessarily imply that $\phi_{n+1} \geqslant \phi_n$ for all values of x and n in question†. We can, however, construct a new scale of functions ψ_n such that

(*a*) ψ_n is identical with ϕ_n for all values of x from a certain value x_n onwards (x_n, of course, depending upon n);

(*b*) $\psi_{n+1} \geqslant \psi_n$ for all values of x and n.

For suppose that we have constructed such a scale up to its nth term ψ_n. Then it is easy to see how to construct ψ_{n+1}. Since $\phi_{n+1} \succ \phi_n$, $\phi_n \sim \psi_n$, it follows that $\phi_{n+1} \succ \psi_n$, and so $\phi_{n+1} \geqq \psi_n$ from a certain value of x (say x_{n+1}) onwards. For $x \geqslant x_{n+1}$ we take $\psi_{n+1} = \phi_{n+1}$. For $x < x_{n+1}$ we give ψ_{n+1} a value equal to the greater of the values of ϕ_{n+1}, ψ_n. Then it is obvious that ψ_{n+1} satisfies the conditions (*a*) and (*b*).

Now let $$f(n) = \psi_n(n).$$

* This is the theorem usually called the 'Theorem of Paul du Bois-Reymond'; see for example Borel, 1, 113. Actually the theorem first proved explicitly by du Bois-Reymond was the corresponding theorem for descending scales (Theorem 3, § 2.4). See du Bois-Reymond, 4, 365.

† $\phi_{n+1} \succ \phi_n$ implies $\phi_{n+1} > \phi_n$ for sufficiently large values of x, say for $x > x_n$. But x_n may tend to infinity with n. Thus $x_n = n+1$ if $\phi_n = x^n/n!$

From $f(n)$ we can deduce a continuous and increasing function $f(x)$, such that

$$\psi_n(x) < f(x) < \psi_{n+1}(x)$$

for $n < x < n+1$, by joining the points $(n, \psi_n(n))$ by straight lines or suitably chosen arcs of curves. Then

$$f/\psi_n > \psi_{n+1}/\psi_n$$

for $x > n+1$, and so $f \succ \psi_n$; therefore $f \succ \phi_n$, and the theorem is proved.

It is perhaps worth while to call attention explicitly to a small point that has sometimes been overlooked*. It is not always the case that the use of straight lines will ensure

$$f(x) > \psi_n(x)$$

for $x > n$ (see, for example, Fig. 2, where the dotted line represents an appropriate arc).

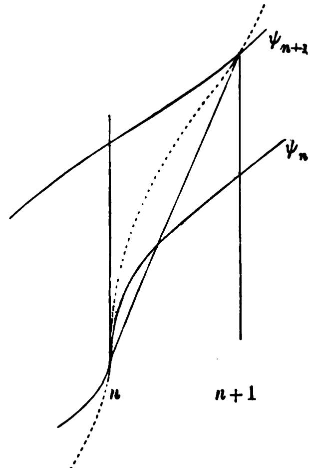

FIG. 2

The proof which precedes may be made more general by taking $f(n) = \psi_\nu(n)$, where ν is an integer depending upon n and tending steadily to infinity with n.

2.22. The second proof of du Bois-Reymond's Theorem proceeds on entirely different lines. We can always choose positive coefficients a_n so that

$$f(x) = \sum_1^\infty a_n \psi_n(x)$$

is convergent for all values of x. This will certainly be the case, for instance, if

$$1/a_n = \psi_1(1)\,\psi_2(2)\,\ldots\,\psi_n(n).$$

For then, if ν is any integer greater than x, $\psi_n(x) < \psi_n(n)$ for $n \geqq \nu$, and the series will certainly be convergent if

$$\sum_\nu^\infty \frac{1}{\psi_1(1)\psi_2(2)\ldots\psi_{n-1}(n-1)}$$

is convergent, as is obvious.

Also $$f(x)/\psi_n(x) > a_{n+1}\psi_{n+1}(x)/\psi_n(x) \rightarrow \infty,$$

so that $f \succ \phi_n$ for all values of n.

2.31. Suppose, *e.g.*, that $\phi_n = x^n$. If we restrict ourselves to values of x greater than 1, we may take $\psi_n = \phi_n = x^n$. The first method of construction would naturally lead to $$f = n^n = e^{n \log n},$$

* Borel, 1, 114; 5, 25.

or $f=\nu^n$, where ν is defined as at the end of § 2.21, and each of these functions has an increase greater than that of any power of n. The second method gives

$$f(x)=\sum_1^\infty \frac{x^n}{1^1 2^2 3^3 \dots n^n}.$$

It is known* that when x is large the order of magnitude of this function is roughly the same as that of

$$e^{\frac{1}{2}(\log x)^2/\log\log x}.$$

As a matter of fact it is by no means necessary, in general, in order to ensure the convergence of the series by which $f(x)$ is defined, to suppose that a_n decreases so rapidly. It is very generally sufficient to suppose $1/a_n=\phi_n(n)$: this is always the case, for example, if $\phi_n(x)=\{\phi(x)\}^n$, as the series

$$\sum \left\{\frac{\phi(x)}{\phi(n)}\right\}^n$$

is always convergent. This choice of a_n would, when $\phi=x$, lead to

$$f(x)=\sum\left(\frac{x}{n}\right)^n \sim \sqrt{\left(\frac{2\pi x}{e}\right)}\, e^{x/e}\dagger.$$

But the simplest choice here is $1/a_n=n\,!$, when

$$f(x)=\sum\frac{x^n}{n\,!}=e^x-1\sim e^x;$$

it is naturally convenient to disregard the irrelevant term -1.

2.32. We can always suppose, if we please, that $f(x)$ is defined by a power series $\sum a_n x^n$ convergent for all values of x, in virtue of a theorem of Poincaré's‡ which is of sufficient intrinsic interest to deserve a formal statement and proof.

Theorem 2. *Given any continuous increasing function $\phi(x)$, we can always find an integral function $f(x)$ (i.e. a function $f(x)$ defined by a power series $\sum a_n x^n$ convergent for all values of x) such that $f(x)\succ\phi(x)$.*

The following simple proof is due to Borel§.

Let $\Phi(x)$ be any function (such as the square of ϕ) such that $\Phi\succ\phi$. Take an increasing sequence of positive numbers a_n such that $a_n\to\infty$, and another sequence of numbers b_n such that

$$a_1<b_2<a_2<b_3<a_3<\dots\dots$$

We can then choose a sequence of positive integers ν_n so that (i) $\nu_{n+1}>\nu_n$ and (ii)

$$\left(\frac{a_n}{b_n}\right)^{\nu_n}>\Phi(a_{n+1}).$$

Now let

$$f(x)=\sum\left(\frac{x}{b_n}\right)^{\nu_n}.$$

* Hardy, **6**. See also § 6.3.

† See Lindelöf, **2**, 41 and **3**; le Roy, **1**; and § 6.3.

‡ Poincaré, **1**, 214.

§ Borel, **4**, 27.

This series is convergent for all values of x; for the nth root of the nth term is not greater (when $b_n > x$) than x/b_n, and so tends to zero. Also

$$f(x) > \left(\frac{a_n}{b_n}\right)^{\nu_n} > \Phi(a_{n+1}) > \Phi(x),$$

if $a_n \leqq x < a_{n+1}$, and so for all values of x greater than a_1; so that $f \succ \phi$.

2.4. So far we have confined our attention to ascending scales, such as $x, x^2, x^3, \dots, x^n, \dots$ or (x^n); but it is obvious that we may consider in a similar manner *descending* scales such as $x, \sqrt{x}, \sqrt[3]{x}, \dots, \sqrt[n]{x}, \dots$ or $(\sqrt[n]{x})$. It is very generally (though not always) true that if (ϕ_n) is an ascending scale, and ψ denotes the function inverse to ϕ, then (ψ_n) is a descending scale.

If $\phi > \bar{\phi}$ for all values of x (or all values greater than some definite value), then a glance at Fig. 3 is enough to show that, if ψ and $\bar{\psi}$ are the functions inverse to ϕ and $\bar{\phi}$, then $\psi < \bar{\psi}$ for all values of x (or all values greater than some definite value). We have only to remember that the graph of ψ may be obtained from that of ϕ by looking at the latter from a different point of view (interchanging the parts of x and y). But it is not true that $\phi \succ \bar{\phi}$ involves $\psi \prec \bar{\psi}$. Thus $e^x \succ e^x/x$. The function inverse to e^x is $\log x$: the function inverse to e^x/x is obtained by solving the equation $x = e^y/y$ with respect to y. This equation gives

$$y = \log x + \log y,$$

and it is easy to see that $y \sim \log x$.

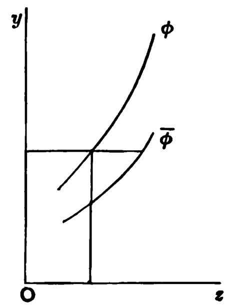

FIG. 3

Theorem 3. *Given a scale of increasing functions ϕ_n such that*

$$\phi_1 \succ \phi_2 \succ \phi_3 \succ \dots \succ 1,$$

we can find an increasing function f such that $\phi_n \succ f \succ 1$ for all values of n.

The proof of this theorem, which is in principle the same as the first proof (§ 2.21) of Theorem 1, may be left to the reader.

2.5. The following extensions of Theorems 1 and 3 are due to du Bois-Reymond, Pincherle, and Hadamard*.

Theorem 4. *Given* $\phi_1 \prec \phi_2 \prec \phi_3 \prec \dots \prec \phi_n \prec \dots \prec \Phi$, *we can find f so that $\phi_n \prec f \prec \Phi$ for all values of n.*

* du Bois-Reymond, **7**; Hadamard, **2**; Pincherle, **1**.

Theorem 5. *Given* $\psi_1 \succ \psi_2 \succ \psi_3 \succ \ldots \succ \psi_n \succ \ldots \succ \Psi$, *we can find* f *so that* $\psi_n \succ f \succ \Psi$ *for all values of* n.

Theorem 6. *Given an ascending sequence* (ϕ_n) *and a descending sequence* (ψ_p) *such that* $\phi_n \prec \psi_p$ *for all values of* n *and* p, *we can find* f *so that*

$$\phi_n \prec f \prec \psi_p$$

for all values of n *and* p.

To prove Theorem 4 we have only to observe that

$$\Phi/\phi_1 \succ \Phi/\phi_2 \succ \ldots \succ \Phi/\phi_n \succ \ldots \succ 1,$$

and to construct (as we can in virtue of Theorem 3) a function F which tends to infinity more slowly than any of the functions Φ/ϕ_n. Then

$$f = \Phi/F$$

is a function such as is required. Similarly for Theorem 5. Theorem 6 requires a little more attention.

In the first place, we may suppose that $\phi_{n+1} > \phi_n$ for all values of x and n: for if this is not so we can modify the definitions of the functions ϕ_n as in § 2.21. Similarly we may suppose $\psi_{p+1} < \psi_p$ for all values of x and p.

Secondly, we may suppose that, if x is fixed, $\phi_n \to \infty$ as $n \to \infty$, and $\psi_p \to 0$ as $p \to \infty$. For if this is not true of the functions given, even when their definitions are modified as above, we may replace the modified ϕ_n and ψ_p by $\Phi_n = 2^n \phi_n$ and $\Psi_p = 2^{-p} \psi_p$; and then $\Phi_n > 2^n \phi_1$, $\Psi_p < 2^{-p} \psi_1$, so that $\Phi_n \to \infty$ when $n \to \infty$ and $\Psi_p \to 0$ when $p \to \infty$.

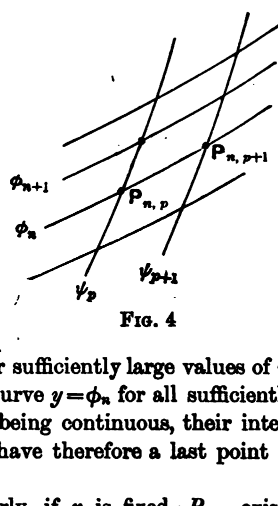

FIG. 4

Since $\psi_p \succ \phi_n$ but, for any given x, $\psi_p < \phi_n$ for sufficiently large values of n, it is clear that the curve $y = \psi_p$ intersects the curve $y = \phi_n$ for all sufficiently large values of n (say for $n > n_p$). The curves being continuous, their intersections form a closed set of points; and they have therefore a last point of intersection, which we denote by $P_{n,p}$.

If p is fixed, $P_{n,p}$ exists for $n > n_p$; similarly, if n is fixed, $P_{n,p}$ exists for $p > p_n$. And as either n or p increases, so do both the ordinate and the abscissa of $P_{n,p}$. The curve $y = \psi_p$ contains all the points $P_{n,p}$ for which p has a fixed value, and $y = \phi_n$ contains all the points for which n has a fixed value.

It is clear that, in order to define a function f which tends to infinity more rapidly than any ϕ_n and less rapidly than any ψ_p, all that we have to do is to draw a curve, making everywhere a positive acute angle with each of the axes of coordinates, and crossing all the curves $y = \phi_n$ from below to above, and all the curves $y = \psi_p$ from above to below.

Choose a positive integer N_p, corresponding to each value of p, such that (i) $N_p > n_p$ and (ii) $N_p \to \infty$ as $p \to \infty$. Then $P_{N_p, p}$ exists for each value of p. And it is clear that we have only to join the points $P_{N_1, 1}$, $P_{N_2, 2}$, $P_{N_3, 3}$, ... by straight lines or other suitably chosen arcs of curves in order to obtain a curve which fulfils our purpose. The theorem is therefore established.

2.61. There are further interesting developments concerning scales of infinity due to Pincherle*.

We have defined $f \succ \phi$ to mean $f/\phi \to \infty$, or, what is the same thing,

(2.611) $$\log f - \log \phi \to \infty.$$

We might equally well have defined $f \succ \phi$ to mean

(2.612) $$F(f) - F(\phi) \to \infty,$$

where $F(x)$ is any function which tends steadily to infinity with x (*e.g.* x, e^x). Let us say that if (2.612) holds then

(2.613) $$f \succ \phi \ (F),$$

so that $f \succ \phi$ is equivalent to $f \succ \phi$ ($\log x$). Similarly we define $f \prec \phi \ (F)$ to mean that $F(f) - F(\phi) \to -\infty$, and $f \asymp \phi \ (F)$ to mean that $F(f) - F(\phi)$ is bounded. Thus

$$x + \log x \asymp x, \quad x + \log x \succ x \ (x),$$
$$x + 1 \asymp x \ (x), \quad x + 1 \succ x \ (e^x),$$

since $e^{x+1} - e^x = (e - 1) e^x \to \infty$.

It is clear that, the more rapid the increase of F, the more likely is it to discriminate between the rates of increase of two given functions f and ϕ. More precisely, *if*

$$f \succ \phi \ (F),$$

and $\bar{F} = FF_1$, *where* F_1 *is any increasing function, then*

$$f \succ \phi \ (\bar{F}).$$

For

$$\bar{F}(f) - \bar{F}(\phi) = F(f) F_1(f) - F(\phi) F_1(\phi) > \{F(f) - F(\phi)\} F_1(\phi) \to \infty.$$

2.62. The substance of the following theorems is due in part to du Bois-Reymond and in part to Pincherle†.

Theorem 7. *However rapid the increase of f, as compared with that of ϕ, we can choose F so that $f \asymp \phi \ (F)$.*

Theorem 8. *If $f - \phi$ is positive for $x > x_0$, we can choose F so that $f \succ \phi \ (F)$.*

* Pincherle, 1.

† du Bois-Reymond, 4; Pincherle, 1.

Theorem 9. *If $f \geqq \phi$ and $f-\phi$ is monotonic for $x > x_0$, and $f \asymp \phi\ (F)$, however great be the increase of F, then $f = \phi$ from a certain value of x onwards.*

(1) If $f \succ \phi$, we may regard f as an increasing function of ϕ, say

$$f = \theta(\phi),$$

where $\theta(x) \succ x$. We can choose a constant g greater than 1, and then choose X so that $\theta(x) > gx$ for $x > X$. Let a be any number greater than X, and let

$$a_1 = \theta(a),\ a_2 = \theta(a_1),\ a_3 = \theta(a_2), \ldots.$$

Then (a_n) is an increasing sequence, and $a_n \to \infty$, since $a_n > g^n a$.

We can now construct an increasing function F such that

$$F(a_n) = \tfrac{1}{2}nK,$$

where K is a constant. Then if $a_{\nu-1} \leqq x \leqq a_\nu$, $a_\nu \leqq \theta(x) \leqq a_{\nu+1}$, and

$$F\{\theta(x)\} - F(x) \leqq F(a_{\nu+1}) - F(a_{\nu-1}) = K.$$

Thus $F(f) - F(\phi)$ remains less than a constant, and Theorem 7 is established.

(2) Let $f - \phi = \lambda$, so that $\lambda > 0$. If λ, as x increases, remains greater than a constant K, then

$$e^f - e^\phi > (e^K - 1)\, e^\phi \to \infty,$$

so that we may take $F(x) = e^x$.

If it is not true that $\lambda \geqq K$, the lower bound of λ is zero. Let $\lambda_1(x)$ be defined as the lower bound of $\lambda(\xi)$ for $\xi \leqq x$. Then λ_1 tends steadily to zero as $x \to \infty$, and $\lambda_1 \leqq \lambda$. We may also regard λ_1 as a steadily decreasing function of ϕ, say $\lambda_1 = \mu(\phi)$.

Let $\varpi(\phi)$ be an increasing function of ϕ such that $\mu\varpi \succ 1$. Then if

$$F(\phi) = \int^\phi \varpi(t)\, dt,$$

$$F(f) - F(\phi) = \int_\phi^{\phi+\lambda} \varpi\, dt \geqq \int_\phi^{\phi+\mu(\phi)} \varpi\, dt > \mu(\phi)\, \varpi(\phi) \succ 1,$$

and $F(x)$ fulfils the requirement of Theorem 8. Finally, Theorem 9 is an obvious corollary of Theorem 8.

The three theorems which follow are of the same character as those which we have just proved. The reader will find it instructive to deduce them, or prove them independently.

Theorem 10. *However great be the increase of f, as compared with that of ϕ, we can determine an increasing function F such that $F(f) \asymp F(\phi)$.*

Theorem 11. *If $f - \phi$ is positive for $x > x_0$, we can determine an increasing function F such that $F(f) \succ F(\phi)$.*

Theorem 12. *If $f \gtrsim \phi$ and $f-\phi$ is monotonic for $x > x_0$, and $F(f) \asymp F(\phi)$, however great the increase of F, then $f=\phi$ from a certain value of x onwards.*

To these he may add the theorem (analogous to that proved at the end of § 2.61) that *$f \succ \phi$ involves $F(f) \succ F(\phi)$ if $\log F(x)/\log x$ is an increasing function* (a condition which is roughly equivalent to $F \succ x$).

2.63. Let us consider some examples of the theorems of the last paragraph.

(i) Let $f=x^m$ $(m>1)$ and $\phi=x$. Then, following the argument of § 2.62 (1), we have $\theta(\phi)=\phi^m$. We may take

$$a=e,\ a_1=e^m,\ a_2=e^{m^2},\ \ldots,\ a_n=e^{m^n},\ \ldots,$$

and we have to define F so that

$$F(e^{m^n})=\tfrac{1}{2}nK.$$

The most natural solution of this equation is

$$F(x)=\frac{K\log\log x}{2\log m}.$$

And in fact

$$F(x^m)-F(x)=\frac{K}{2\log m}\{\log(m\log x)-\log\log x\}=\tfrac{1}{2}K,$$

so that $x^m \asymp x\ (F)$.

(ii) Let $f=e^x+e^{-x}$, $\phi=e^x$. Following the argument of § 2.62 (2), we find

$$\lambda=e^{-x}=\lambda_1,\quad \mu(\phi)=1/\phi,$$

and we may take $\varpi(\phi)=\phi^{1+a}$ $(a>0)$. This makes $F(\phi)$ a constant multiple of ϕ^{2+a}, and it is easy to verify that

$$(e^x+e^{-x})^k-e^{kx}\to\infty,$$

if $k>2$.

(iii) The relation $F(f)\asymp F(\phi)$ is equivalent to $f\asymp\phi\ (\log F)$. Using the result of (i), we see that $F(x^m)\asymp F(x)$ if $1\prec F\preccurlyeq\log x$. Similarly, using the result of (ii), we see that $F(e^x+e^{-x})\succ F(e^x)$ if $F\succcurlyeq e^{x^k}$ $(k>2)$.

2.7. Before leaving this part of our subject, let us observe that the substance of §§ 2.1—2.5 may be extended to the case in which our symbols $\succ$, etc. are defined by reference to an arbitrary increasing function F. We leave it as an exercise to the reader to effect these extensions.

III

LOGARITHMICO-EXPONENTIAL SCALES: THE FUNDAMENTAL THEOREMS

3.1. THE scales of infinity that are of most practical importance in analysis are those which may be constructed by means of the logarithmic and exponential functions.

We have already seen (§ 1.1) that

$$e^x \succ x^n$$

for any value of n; and from this it follows that

$$\log x \prec x^{1/n}$$

for any value of n*. It is easy to deduce that

$$e^{e^x} \succ e^{x^n},\quad e^{e^{e^x}} \succ e^{e^{x^n}},\ \dots,$$

$$\log\log x \prec (\log x)^{1/n},\quad \log\log\log x \prec (\log\log x)^{1/n},\ \dots.$$

The repeated logarithmic and exponential functions are so important in this subject that it is worth while to adopt a notation for them of a less cumbrous character. We shall write

$$l_1 x = lx = \log x,\quad l_2 x = llx,\quad l_3 x = ll_2 x,\ \dots\dagger,$$

$$e_1 x = ex = e^x,\quad e_2 x = eex,\quad e_3 x = ee_2 x,\ \dots.$$

It is easy, with the aid of these functions, to write down any number of ascending scales, each containing only functions whose increase is greater than that of any function in any preceding scale: for example

$$x,\ x^2,\ \dots,\ x^n,\ \dots;\ e^x,\ e^{2x},\ \dots,\ e^{nx},\ \dots;\ e^{x^2},\ e^{x^3},\ \dots,\ e^{x^n},\ \dots.$$

Among the functions of these scales we can interpolate new functions as freely as we like, using, for instance, such functions as

$$x^{\alpha} e^{\beta x^{\gamma} e^{\delta x^{\epsilon}}},$$

where α, β, γ, δ, ϵ are any positive numbers; and we can construct non-enumerable as well as enumerable scales‡. Similarly we can construct any number of descending scales, each composed of functions whose

* It was pointed out in § 2.4 that $\phi \succ \bar{\phi}$ does not necessarily involve $\psi \prec \bar{\psi}$ (ψ, $\bar{\psi}$ being the functions inverse to ϕ, $\bar{\phi}$). But it does involve $\psi < \bar{\psi}$ for sufficiently large values of x, and therefore $\psi \preccurlyeq \bar{\psi}$. Hence $\phi \succ \phi_n$ (for any n) involves $\psi \preccurlyeq \psi_n$ (for any n) and therefore, if (ψ_n) is a descending scale, as is in this case obvious, $\psi \prec \psi_n$ for any n.

† lx is defined for $x > 0$, $l_2 x$ for $x > 1$, $l_3 x$ for $x > e$, $l_4 x$ for $x > e^e$, and so on.

‡ See § 2.1.

increase is less than that of any functions in any preceding scale: for example

$$lx,\ (lx)^{1/2},\ \ldots,\ (lx)^{1/n},\ \ldots;\ l_2x,\ (l_2x)^{1/2},\ \ldots,\ (l_2x)^{1/n},\ \ldots.$$

Two special scales are of fundamental importance; the ascending scale

$$(E)\qquad x,\ ex,\ e_2x,\ e_3x,\ \ldots,$$

and the descending scale

$$(L)\qquad x,\ lx,\ l_2x,\ l_3x,\ \ldots.$$

These scales mark the *limits* of all logarithmic and exponential scales. It is of course possible, in virtue of the general theorems of §§ 2.1—2.5, to define functions whose increase is more rapid than that of any e_nx or slower than that of any l_nx; but, as we shall see in a moment, this is not possible if we confine ourselves to functions defined by finite and explicit formulae involving only the ordinary functional symbols of elementary analysis.

3.2. We define a *logarithmico-exponential function* (shortly, an *L-function*) as a real one-valued function defined, for all values of x greater than some definite value, by a finite combination of the ordinary algebraic symbols (viz. $+,\ -,\ \times,\ \div,\ \sqrt[n]{\ }$) and the functional symbols $\log(\ldots)$ and $e^{(\ldots)}$, operating on the variable x and on real constants.

It is to be observed that the result of working out the value of the function, by substituting x in the formula defining it, is to be real at all stages of the work. It is important to exclude such a function as

$$\tfrac{1}{2}\{e^{\sqrt{(-x^2)}}+e^{-\sqrt{(-x^2)}}\},$$

which, with a suitable interpretation of the roots, is equal to $\cos x$.

We might generalize our definition by admitting implicit algebraic functions, including, for example, such functions as $e_2\sqrt{(ly)}$, where $y^5+y-x=0$; but this generalization is not particularly interesting.

Theorem 13. *Any L-function is ultimately continuous, of constant sign, and monotonic, and tends, as $x\to\infty$, to infinity, or to zero or to some other definite limit. Further, if f and ϕ are L-functions, one or other of the relations*

$$f\succ\phi,\ f\asymp\phi,\ f\prec\phi$$

holds between them.

We may classify L-functions as follows, by a method similar to that by which Liouville* classified the 'elementary' functions. An L-function is of order 0 if it is purely algebraic; of order 1 if the functional

* Liouville, **1**; Watson, **1**, 111. See also Hardy, **2**; this tract contains fuller references to Liouville's memoirs.

symbols $l(\dots)$ and $e(\dots)$ which occur in it operate only on algebraic functions; of order 2 if they operate only on algebraic functions or L-functions of order 1; and so on. Thus

$$e^{x^2}+\sqrt{(\log x)},\quad x^{\sqrt{2}}=e^{\sqrt{2}\log x},\quad x^{x^x}=e^{\log x\, e^{x\log x}}$$

are of orders 1, 2, and 3 respectively. As the results stated in the theorem are true of algebraic functions, it is sufficient to prove that, if true of L-functions of order $n-1$, they are true of L-functions of order n.

It should be observed that an L-function of order n may always be expressed as a function of any higher order; thus $x=e(lx)=e_2(l_2x)=\dots$. We need not suppose that our functions are always expressed in the simplest possible form. In Liouville's work it is essential to assume that an 'elementary function of order n' cannot be expressed as a function of lower order; but no such hypothesis is necessary here.

The following additional definitions will be found useful. We shall say that f_n, an L-function of order n, is *integral* if it is of the form

$$\Sigma\rho_{n-1}e\sigma_{n-1}(l\tau_{n-1}^{(1)})^{\kappa_1}(l\tau_{n-1}^{(2)})^{\kappa_2}\dots(l\tau_{n-1}^{(h)})^{\kappa_h},$$

where the functions with suffix $n-1$ are L-functions of order $n-1$ and $\kappa_1, \kappa_2, \dots, \kappa_h$ are positive integers. We call $\kappa_1+\kappa_2+\dots+\kappa_h$ the *logarithmic degree*, or, simply, the *degree*, of the typical term of f_n; and, if λ is the greatest value of $\kappa_1+\kappa_2+\dots+\kappa_h$, we say that f_n is *of logarithmic degree* λ. If the number of terms of degree λ in f_n is μ, we say that f_n is *of logarithmic type* (λ, μ). We denote integral L-functions by the letter M, with or without suffixes, indices, etc.

If an integral L-function is of degree 0, *i.e.* of the form

$$\Sigma\rho_{n-1}e\sigma_{n-1},$$

we shall say that it is *exponential*. If an integral exponential L-function contains ϖ terms, we shall say that it is *of type* ϖ; if $\varpi=1$, we shall say that it is *simply exponential*. Thus $(lx)^2e(e^x lx)$ is a simply exponential L-function of order 2, while $(lx)^2(l_2x)^2e(e^x lx)$ is an integral function of order 2, of type (2, 1). We shall in general denote integral exponential L-functions by the letter N.

If f_n is the quotient of two integral functions, *i.e.* of the form M_1/M_2, we shall say that it is *rational*. If M_1 and M_2 are exponential, *i.e.* if f_n is of the form N_1/N_2, we shall say that f_n is a *rational exponential* L-function.

It may be verified immediately that:

(i) The derivative of an L-function of order n is an L-function of order n. In exceptional cases the derivative may be of order $n-1$.

(ii) The derivative of a simply exponential function is a simply exponential function, with the same exponential factor.

(iii) The derivative of an integral exponential function of type ϖ is in general an integral exponential function of type ϖ. If one of the terms of the original function is a constant, the derivative is of type $\varpi - 1$.

(iv) The derivative of an integral function of logarithmic type (λ, μ) is in general a function of the same type. If the exponential factor $\rho_{n-1} e\sigma_{n-1}$ of one of the terms of degree λ is a constant, then the derivative is of type $(\lambda, \mu - 1)$; if $\mu = 1$, the derivative is of degree $\lambda - 1$.

3.3. We can simplify the induction required for the proof of Theorem 13 by two preliminary remarks.

(i) If f is an L-function of order n, so is its derivative f'. Hence, if every such function is ultimately continuous and of constant sign, every such function is ultimately monotonic.

(ii) If f and ϕ are L-functions of order n, so is f/ϕ. Hence, if every such function is ultimately monotonic, f/ϕ must tend to infinity or a limit, and one of the relations $\succ$, $\asymp$, $\prec$ holds between the functions.

It is therefore sufficient to prove that *if Theorem* 13 *is true for functions of order* $n-1$, *then any function of order* n *is ultimately continuous and of constant sign.*

We prove this first when f_n is an integral exponential function. The result is obvious when f_n is of type 1 (*i.e.* when f_n is a simply exponential function $\rho_{n-1} e\sigma_{n-1}$). Let us then assume it true for functions of type $\varpi - 1$; and let

$$f_n = \Sigma \rho_{n-1} e\sigma_{n-1}$$

be of type ϖ. If $\rho_{n-1} e\sigma_{n-1}$ is any one of the terms of f_n, the function

$$F_n = f_n/(\rho_{n-1} e\sigma_{n-1})$$

is of type ϖ, with one term a constant (unity); and so, by § 3.2 (iii), F_n' is of type $\varpi - 1$. Hence F_n' is ultimately continuous and of constant sign; and so the same is true of F_n, and therefore of f_n.

We prove next that the result is true when f_n is any integral function of order n. Suppose that

$$f_n = \Sigma \rho_{n-1} e\sigma_{n-1} (l\tau_{n-1}^{(1)})^{\kappa_1} (l\tau_{n-1}^{(2)})^{\kappa_2} \dots (l\tau_{n-1}^{(h)})^{\kappa_h}$$

is of logarithmic type (λ, μ). The result has been proved true when $\lambda = 0$. Hence it is enough to prove

(i) that, if true for functions of logarithmic degree $\lambda - 1$, it is true for functions of degree λ and type $(\lambda, 1)$;

(ii) that, if true for functions of type $(\lambda, \mu-1)$, it is true for functions of type (λ, μ).

Suppose that the typical term written above in the expression of f_n is one of the terms of degree λ, and let $F_n = f_n/(\rho_{n-1} e\sigma_{n-1})$ as before. Then, by § 3.2 (iv), F_n' is of type $(\lambda, \mu-1)$, unless $\mu=1$, when it is of degree $\lambda-1$. Hence, whichever of the inductions (i), (ii) we are engaged in proving, F_n' is ultimately continuous and of constant sign; and we deduce as before that the same is true of f_n.

We are now in a position to complete the proof of the theorem. Any L-function f_n is of the form

$$f_n = A\{e\phi_{n-1}^{(1)}, e\phi_{n-1}^{(2)}, \dots, e\phi_{n-1}^{(r)}, l\psi_{n-1}^{(1)}, \dots, l\psi_{n-1}^{(s)}, \chi_{n-1}^{(1)}, \dots, \chi_{n-1}^{(t)}\}$$
$$= A(z_1, z_2, \dots, z_q),$$

say, where $q = r+s+t$ and A is an algebraic function of its arguments. There is therefore an identical relation

$$F(x, y) = M_0 y^p + M_1 y^{p-1} + \dots + M_p = 0,$$

where $y = f_n$ and the coefficients $M_1, M_2, \dots, M_p$ are integral L-functions of order n. The derivatives of these coefficients are also integral. It therefore follows from what has already been proved that

$$F_x = \frac{\partial F}{\partial x} = \Sigma \frac{dM_i}{dx} y^{p-i}, \quad F_y = \frac{\partial F}{\partial y} = \Sigma (p-i) M_i y^{p-i-1},$$

considered as functions of the two variables x, y, are continuous for all sufficiently large values of x and for all values of y.

Let ξ, η be a pair of values of x and y satisfying the equation $F=0$. Then, if only ξ is large enough, F_y cannot vanish for $x=\xi$, $y=\eta$. For, if F and F_y both vanish for $x=\xi$, $y=\eta$, the eliminant of y between $F=0$ and $F_y=0$ vanishes for $x=\xi$. But this eliminant is plainly an integral L-function of order n, and so cannot vanish for values of x surpassing all limit. It follows, by the fundamental theorem concerning implicit functions*, that f_n is an ultimately continuous function of x. Finally, f_n is ultimately of constant sign. For $f_n = 0$ involves $M_p = 0$, and we have already seen that it is impossible that this equation should be satisfied for values of x surpassing all limit. This completes the proof of the theorem.

3.4. The limits of the increase of L-functions. The increase of an increasing L-function is subject to limitations of rapidity or slowness. We may say roughly that *an L-function cannot increase more rapidly than any exponential or more slowly than any logarithm;* if f is

* Goursat, 1 (1), 81, 94; Hardy, 1, 192; Young, 1.

any L-function, we can determine k so that $f \prec e_k x$, and if f is any L-function which tends to infinity, we can determine k so that $f \succ l_k x$.

More precisely, we have the two following theorems:

Theorem 14. *An L-function of order n cannot satisfy*

$$f_n \succ e_n(x^{\Delta}).$$

Theorem 15. *An L-function of order n cannot satisfy*

$$1 \prec f_n \prec (l_n x)^{\delta}.$$

Theorem 14 is very easy to prove. It is plainly sufficient to establish an induction from $n-1$ to n.

Any function of order n is an algebraical function of certain arguments $e\phi_{n-1}, \ldots, l\psi_{n-1}, \ldots, \chi_{n-1}, \ldots$, the increase of any one of which is *ex hypothesi* less than that of

$$e(e_{n-1}x^K) = e_n(x^K)$$

for some value of K. Hence the increase of the function is less than that of

$$(e_n x^K)^{K_1}$$

for some values of K and K_1; and so less than that of $e_n(x^{K_2})$ for some value of K_2. Thus the theorem is established.

The proof of Theorem 15 is considerably more troublesome, and, though it presents no particular difficulty of principle, is too long to be inserted here*. It is included in a more precise theorem, viz.

Theorem 16. *If f is an L-function of order n, and*

$$1 \prec f \prec (l_{n-1}x)^{\delta},$$

then

$$f \asymp (l_n x)^h,$$

where h is rational.

3.5. Let f and ϕ be any two L-functions which tend to infinity with x, and let α be any positive number. Then one of the three relations

$$f \succ \phi^{\alpha}, \quad f \asymp \phi^{\alpha}, \quad f \prec \phi^{\alpha}$$

must hold between f and ϕ; and the second can hold for at most one value of α. If the first holds for any α it holds for any smaller α; and if the last holds for any α it holds for any greater α.

Then there are three possibilities. Either the first relation holds for every α; then

$$f \succ \phi^{\Delta}.$$

Or the third holds for every α; then

$$f \prec \phi^{\delta}.$$

* The details of the proof will be found in Hardy, 9, 65—72.

Or the first holds for some values of α and the third for others; and then there is a value a of α which divides the two classes of values of α, and we may write

$$f = \phi^a f_1,$$

where $\phi^{-\delta} \prec f_1 \prec \phi^\delta$. We shall find this result very useful in the sequel.

3.6. It is possible to classify the possible modes of increase of L-functions of given order much more precisely. Thus we have:

Theorem 17. *An L-function of order* 1, *which tends to infinity with x, is of one of the forms*

$$e^{Ax^s(1+\epsilon)}, \quad Ax^s(\log x)^t(1+\epsilon),$$

where s and t are rational.

Theorem 18. *An L-function of order* 2, *which tends to infinity with x, is of one of the forms*

$$e^{e^{Ax^s(1+\epsilon)}}, \quad e^{Ax^s(lx)^t(1+\epsilon)}, \quad x^s e^{A(lx)^t(1+\epsilon)}, \quad Ax^s(lx)^t(l_2x)^u(1+\epsilon),$$

where s, t, and u are rational.

The functions

$$e_2x, \quad l_2x, \quad x^x, \quad x^{\sqrt{2}}, \quad e^{\sqrt{(lx)}}$$

each increase in a manner which differs from either of those of Theorem 17. They are therefore not expressible as L-functions of order lower than 2. Similarly l_3x, $(lx)^{\sqrt{2}}$, or $e^{\sqrt{(l_2x)}}$ are not expressible as L-functions of order lower than 3. No L-function of order 1 can satisfy

$$x^\Delta \prec f \prec e^{x^\delta},$$

and no L-function of order 2 can satisfy either of

$$ex^\Delta \prec f \prec e_2x^\delta, \quad e(lx)^\Delta \prec f \prec ex^\delta.$$

The reader will find detailed proofs of these theorems in a memoir by the author*.

* Hardy, 9.

IV

SPECIAL PROBLEMS CONNECTED WITH LOGARITHMICO-EXPONENTIAL SCALES

4.1. The functions $e_r(l_s x)^\mu$. We have agreed to express the fact that, however large be a and however small be b, x^a has an increase less than that of e^{x^b}, by

(4.11) $$x^\Delta \prec e^{x^\delta}.$$

Let us endeavour to find a function f such that

(4.12) $$x^\Delta \prec f \prec e^{x^\delta}.$$

If $\phi_1 \succ \phi_2$, $e^{\phi_1} \succ e^{\phi_2}$. Thus (4.12) will certainly be satisfied if

$$\log x \prec \log f \prec x^\delta.$$

Hence a solution of our problem is given by

$$f = e^{(\log x)^\beta} \quad (\beta > 1).$$

Similarly we can prove that

$$f = e^{(\log x)^\alpha} \quad (0 < \alpha < 1)$$

satisfies $$(\log x)^\Delta \prec f \prec x^\delta.$$

It will be convenient to write

$$e_0 x \equiv l_0 x \equiv x,$$

α for a positive number less than 1, β for a positive number greater than 1, and γ for any positive number; and then we have the relations

(4.13) $$e_0(l_1 x)^\gamma \prec e_1(l_1 x)^\alpha \prec e_0(l_0 x)^\gamma \prec e_1(l_1 x)^\beta \prec e_1(l_0 x)^\gamma.$$

Let us now consider the functions

$$f = e_r(l_s x)^\mu, \quad f' = e_{r'}(l_{s'} x)^{\mu'},$$

where μ, μ' are positive and not equal to 1. If $r = r'$, $f \succ f'$ or $f \prec f'$ according as $s < s'$ or $s > s'$. If $s = s'$, the same relations hold according as $r > r'$ or $r < r'$. If $r = r'$ and $s = s'$, then $f \succ f'$ or $f \prec f'$ according as $\mu > \mu'$ or $\mu < \mu'$. Leaving these cases aside, suppose $s > s'$, $s - s' = \sigma > 0$. Putting $l_{s'} x = y$, we obtain

$$f = e_r(l_\sigma y)^\mu, \quad f' = e_{r'} y^{\mu'}.$$

If $r \leqq r'$, it is clear that $f \prec \phi$. If $r > r'$, let $r - r' = \rho$; then

$$l_r f = (l_\sigma y)^\mu, \quad l_r f' = l_\rho y^{\mu'} \asymp l_\rho y:$$

if $\rho > 1$ the symbol $\asymp$ may be replaced by $\sim$. If $\sigma > \rho$, $l_r f \prec l_r f'$ and so $f \prec f'$. If $\sigma < \rho$, $f \succ f'$. If $\sigma = \rho$, $f \succ f'$ or $f \prec f'$ according as $\mu > 1$ or $\mu < 1$. Thus

$$f \succ f' \; (r - s > r' - s'), \quad f \prec f' \; (r - s < r' - s'),$$

while if $r - s = r' - s'$, $f \succ f'$ or $f \prec f'$ according as $\mu > 1$ or $\mu < 1$, μ being the exponent of the logarithm of higher order which occurs in f or f'.

From this it follows that

$$\ldots e_1(l_2x)^\alpha \prec e_0(l_1x)^\gamma = (lx)^\gamma \prec e_1(l_2x)^\beta \prec e_2(l_3x)^\beta \prec \ldots,$$

$$\ldots \prec e_2(l_2x)^\alpha \prec e_1(l_1x)^\alpha \prec e_0(l_0x)^\gamma = x^\gamma \prec e_1(l_1x)^\beta \prec \ldots,$$

$$\ldots \prec e_3(l_2x)^\alpha \prec e_2(l_1x)^\alpha \prec e_1(l_0x)^\gamma = ex^\gamma \prec e_2(l_1x)^\beta \prec \ldots.$$

These relations enable us to interpolate to any extent among what we may call the fundamental logarithmico-exponential orders of infinity, viz. $(l_kx)^\gamma$, x^γ, e_kx^γ. Thus

$$e^{(lx)^\beta},\quad e^{e^{(llx)^\beta}},\ \ldots \quad (\beta > 1),$$

and

$$e^{e^{(lx)^\alpha}},\quad e^{e^{e^{(llx)^\alpha}}},\ \ldots \quad (0 < \alpha < 1),$$

are two scales, the first rising from above x^γ, the second falling from below ex^γ, and never overlapping.

These scales, and the analogous scales which can be interpolated between other pairs of the fundamental logarithmico-exponential orders, possess another interesting property. The two scales written above *cover up* (to put it roughly) *the whole interval between x^γ and ex^γ, so far as L-functions are concerned*: that is to say, it is impossible that an L-function f should satisfy

$$f \succ e_r(l_rx)^\beta, \qquad (\textit{every } r),$$

$$f \prec e_{r+1}(l_rx)^\alpha, \qquad (\textit{every } r);$$

and the corresponding pairs of scales lying between $(l_{k+1}x)^\gamma$ and $(l_kx)^\gamma$, or between e_kx^γ and $e_{k+1}x^\gamma$, possess a similar property. This property is analogous to that possessed (Theorems 14 and 15) by the scales (l_rx), (e_rx); viz. that no L-function f can satisfy $f \succ e_rx$, or $1 \prec f \prec l_rx$, for all values of r. A little consideration is all that is needed to render the theorem plausible: for a formal proof we must refer to the memoir quoted on p. 21.

4.21. Successive approximations to a logarithmico-exponential function. Consider such a function as

$$f = \sqrt{x}\,(lx)^2 e^{\sqrt{(lx)}\,(l_2x)^2 e^{\sqrt{(l_3x)}\,(l_3x)^2}}.$$

If we omit one or more of the parts of the expression of f, we obtain another function whose increase differs more or less widely from that of f. The question arises which parts are of the greatest and which of the least importance, *i.e.* which are the parts whose omission affects the increase of f most or least fundamentally.

Taking logarithms we find

$$lf = \tfrac{1}{2}lx + \sqrt{(lx)}\,(l_2x)^2 e^{\sqrt{(l_3x)}\,(l_3x)^2} + 2l_2x, \tag{4.211}$$

the three terms being arranged in order of importance. Again

$$l_2f = l_2x - l2 + \epsilon, \quad l_3f = l_3x + \epsilon.$$

If we neglect the ϵ's in these equations, we deduce the approximations

$$(1)\ f = x, \qquad (2)\ f = \sqrt{x}.$$

By neglecting the last term in the equation (4.211) we obtain the much closer approximation

$$(6)\ f=\sqrt{x}e^{\sqrt{(lx)}(l_2x)^2}e^{\sqrt{(l_2x)}(l_3x)^2}.$$

In order to obtain a more complete series of approximations, we must replace the equation (4.211) by a series of approximate equations. Now if

$$\phi=\sqrt{(lx)}\,(l_2x)^2\,e^{\sqrt{(l_2x)}(l_3x)^2},$$

we have

$$l\phi=\tfrac{1}{2}l_2x+\sqrt{(l_2x)}\,(l_3x)^2+2l_3x,$$

$$l_2\phi=l_3x-l2+\epsilon,\quad l_3\phi=l_4x+\epsilon.$$

Hence we obtain (0) $\phi=lx$, (3) $\phi=\sqrt{(lx)}$, and (5) $\phi=\sqrt{(lx)}\,e^{\sqrt{(l_2x)}(l_3x)^2}$ as approximations to the increase of ϕ: of these, however, the first is valueless, inasmuch as it would make ϕ preponderate over the first term on the right hand side of (4.211).

A similar argument, applied to the function $e^{\sqrt{(l_2x)}(l_3x)^2}$, leads us to interpolate (4) $\phi=\sqrt{(lx)}\,e^{\sqrt{(l_2x)}}$ between (3) and (5). We can now, by substituting these approximations to ϕ in (4.211), deduce a complete system of closer and closer approximations to the increase of f, viz.

$$(1)\ x,\quad (2)\ \sqrt{x},\quad (3)\ \sqrt{x}\,e^{\sqrt{(lx)}},\quad (4)\ \sqrt{x}\,e^{\sqrt{(lx)}\,e^{\sqrt{(l_2x)}}},$$

$$(5)\ \sqrt{x}\,e^{\sqrt{(lx)}\,e^{\sqrt{(l_2x)}(l_3x)^2}},\quad (6)\ \sqrt{x}\,e^{\sqrt{(lx)}\,(l_2x)^2\,e^{\sqrt{(l_2x)}(l_3x)^2}}.$$

This order corresponds to the order of importance of the various parts of the expression of f.

4.22. Legitimate and illegitimate forms of approximation to a logarithmico-exponential function. In applications of this theory, such as occur, for instance, in the theory of integral functions, we are continually meeting such equations as

$$(4.221)\qquad f=(1+\epsilon)\,e^{x^a},\quad f=e^{(1+\epsilon)\,x^a},\quad f=e^{x^{a+\epsilon}}\quad (a>0).$$

It is important to have clear ideas as to the degree of accuracy of such representations of f. The simplest method is to take logarithms repeatedly, as in § 4.21.

In the first example the term ϵ does not affect the increase of f: we have $f\sim ex^a$. This is not true in the second; but $lf\sim x^a$, so that the term ϵ does not affect the increase of lf; while in the third this is not true, though $llf\sim a$. Of the three formulae the first gives the most, and the last the least, information as to the increase of f.

Such a formula as

$$(4.222)\qquad f=xe^{(1+\epsilon)\,x^a}$$

would not be a legitimate form of approximation at all, since the factor $e\,(\epsilon x^a)$ may well be far more important than the accurate factor x, and (4.222) conveys no more information than the second equation (4.221).

4.3. Attempts to represent orders of infinity by symbols. It is natural to try to devise some simple method of representing orders of infinity by symbols which can be manipulated according to laws resembling as far as possible those of ordinary algebra. Thus Thomae* has proposed to represent the order of infinity of $f=x^a(lx)^{a_1}(l_2x)^{a_2}\ldots$ by

$$Of=a+a_1l_1+a_2l_2+\ldots\dagger,$$

where the symbols l_1, l_2, ... are to be regarded as new units. It is clear that these units cannot, in relation to one another, obey the Axiom of Archimedes‡: however great n, nl_2 cannot be as great as l_1, nor nl_1 as great as 1.

The consideration of a few simple cases is enough to show that any such notation, if it is to be useful, must obey the following laws:

$$\text{(i)}\quad \text{if } f\succ\phi,\quad O(f+\phi)=Of;$$
$$\text{(ii)}\quad O(f\phi)=Of+O\phi;$$
$$\text{(iii)}\quad O\{f(\phi)\}=Of\times O\phi.$$

And Pincherle§ has pointed out that these laws are in any case inconsistent with the maintenance of the laws of algebra in their entirety. Thus if

$$Ox=1,\quad Olx=\lambda,$$

we have, by (iii), $Ollx=\lambda^2$, and by (iii) and (ii),

$$Ol(xlx)=\lambda(1+\lambda)=\lambda+\lambda^2;$$

and on the other hand, by (i),

$$Ol(xlx)=O(lx+llx)=\lambda.$$

Pincherle has suggested another system of notation; but the best yet formulated is Borel's‖. Borel preserves the three laws (i), (ii), (iii), the commutative law of addition, and the associative law of multiplication. But multiplication is no longer commutative, and distributive on one side only¶. He would denote the orders of

$$e^x x^n,\quad x^n(lx)^p,\quad e^{2x},\quad e^{x^2},\quad e^{e^x},\quad e^{\sqrt{lx}},\quad \tfrac{1}{2}x,$$

by

$$\omega+n,\quad n+\frac{p}{\omega},\quad 2\,.\,\omega,\quad \omega\,.\,2,\quad \omega^2,\quad \omega\,.\,\frac{1}{2}\,.\,\frac{1}{\omega},\quad \frac{1}{\omega}\,.\,\frac{1}{2}\,.\,\omega.$$

But little application, however, has yet been found for any such system of notation; and the whole matter appears to be rather of the nature of a mathematical curiosity.

4.41. Functions which do not conform to any logarithmico-exponential scale. We saw in § 1.2 that, given two increasing functions ϕ and ψ such that $\phi\succ\psi$, we can always construct an increasing function f which is, for an infinity of values of x increasing beyond all limit, of

* Thomae, **1**, 144.

† The reader will not confuse this use of the symbol O (which does not extend beyond this paragraph) with that explained in § 1.5.

‡ 'If $x>y>0$, we can find an integer n such that $ny>x$'.

§ Pincherle, **1**.

‖ Borel, **4**, 35 and **5**, 14.

¶ $(a+b)c=ac+bc$, but in general $a(b+c)\neq ab+ac$.

the order of ϕ, and for another infinity of values of x of the order of ψ. The actual construction of such functions by means of explicit formulae we left till later. We shall now consider the matter more in detail, with special reference to the case in which ϕ and ψ are L-functions.

We shall say that f is an *irregularly increasing* function (*fonction à croissance irrégulière*) if we can find two L-functions ϕ and $\psi\,(\phi \succ \psi)$ such that

$$f \geqslant \phi \ \ (x = x_1, x_2, \ldots), \quad f \leqslant \psi \ \ (x = x_1', x_2', \ldots),$$

$x_1, x_2, \ldots$ and $x_1', x_2', \ldots$ being any two indefinitely increasing sequences of values of x. We shall also say that 'the increase of f is irregular' and that 'the logarithmico-exponential scales are *inapplicable to* f'.

The phrase '*fonction à croissance irrégulière*' has been defined by various writers in various senses. Borel* originally defined f to be *à croissance régulière* if

$$e^{x^{a-\delta}} < f < e^{x^{a+\delta}} \quad (x > x_0),$$

or in other words if $llf \asymp lx$.

This definition was of course designed to meet the particular needs of the theory of integral functions; and has been made more precise by Boutroux and Lindelöf†, who use inequalities of the form

$$e^{x^a (lx)^{a_1} \ldots (l_k x)^{a_k - \delta}} < f < e^{x^a (lx)^{a_1} \ldots (l_k x)^{a_k + \delta}}.$$

All functions which are not *à croissance régulière* for these writers are included in our class of irregularly increasing functions.

4.42. The logarithmico-exponential scales may fail to give a complete account of the increase of a function in two different ways. The function may be of irregular increase, as explained above, and the scales *inapplicable*, or on the other hand they may be, not inapplicable, but *insufficient* (*en défaut*). That is to say, although the increase of the function does not oscillate from that of one L-function to that of another, there may be no L-function capable of measuring it. That such functions exist follows at once from the general theorems of § 3.4. Thus we can define a function which tends to infinity more rapidly than any $e_r x$, or more slowly than any $l_r x$; and the increase of such a function is more rapid or slower than that of any L-function. Or again (§ 2.5, Theorem 6) we can define a function whose increase is, for every r, greater than that of $e_r (l_r x)^\beta$ and less than that of $e_{r+1} (l_r x)^\alpha$, if $0 < \alpha < 1 < \beta$; and (§ 4.1) the increase of such a function cannot be equal to that of any L-function.

We shall now discuss some actual examples of functions for which the logarithmico-exponential scales are inapplicable or insufficient.

* Borel, **2**, 108. † Boutroux, **1**; Lindelöf, **2**. See also Blumenthal, **1**, 7.

4.43. Irregularly increasing functions. Functions whose increase is irregular may be constructed in a variety of ways.

(i) Pringsheim* has used, in connection with the theory of the convergence of series, functions of an integral variable n whose increase is irregular. A simple example of such a function is

$$f(n) = 10^{[(\log_{10} n)^{1/\tau}]^\tau} \quad (\tau > 1),$$

where $[x]$ denotes the integral part of x. It is easily verified, for instance, when $\tau = 2$, that the increase of $f(n)$ varies between that of n and that of $n \,.\, 10^{-2\sqrt{(\log_{10} n)}}$.

(ii) A more natural type of function is given by

$$f = \phi \cos^2 \theta + \psi \sin^2 \theta,$$

where ϕ, ψ, θ are increasing L-functions and $\phi \succ \psi \succ 1$. We have to consider what conditions ϕ, ψ, θ must satisfy in order that f may increase steadily with x. That its increase oscillates between that of ϕ and that of ψ is obvious.

Differentiating, we obtain

$$f' = \phi' \cos^2 \theta + \psi' \sin^2 \theta + 2(\psi - \phi)\,\theta' \cos\theta \sin\theta.$$

We shall now assume that (as will be proved in the next chapter) relations between L-functions, involving the symbols $\succ, \ldots$, may be differentiated and integrated. The condition that f' should always be positive is that

$$\phi'\psi' > (\phi - \psi)^2 \theta'^2,$$

or $\phi'\psi' > \phi^2\theta'^2$. Since $\phi' \succ \psi'$, this involves $\phi' \succ \phi\theta'$, or $\log \phi \succ \theta$; and f will certainly be monotonic if

$$\log \phi \succ \theta, \quad \psi' \succ \phi^2\theta'^2/\phi'.$$

These conditions are satisfied, for example, if $\phi = x^\alpha e x^\rho$, $\psi = x^\beta e x^\rho$, $\theta = x$, and $\alpha - 2\rho + 2 < \beta < \alpha$. Changing our notation a little we see that

$$f = (x^{\gamma+\delta} \cos^2 x + x^{\gamma-\delta} \sin^2 x)\, e^{x^\rho}$$

is monotonic if $0 < \delta < \rho - 1$; and the increase of f oscillates between that of $x^{\gamma+\delta} e x^\rho$ and that of $x^{\gamma-\delta} e x^\rho$. Similarly it may be shown that

$$f = (e^{\mu x} \cos^2 x + e^{\nu x} \sin^2 x)\, e^{e^x}$$

is monotonic if $\nu < \mu < \nu + 2$†; and again the increase of f is irregular.

(iii) Borel‡ has shown how, by means of power series, we may define functions which increase steadily with x, while their increase oscillates to an arbitrary extent.

Let $\qquad \phi(x) = \Sigma a_n x^n, \quad \psi(x) = \Sigma b_n x^n$

be two integral functions of x with positive coefficients. The increase

* Pringsheim, **5** and **1**, 373. † Hardy, **3** (1).

‡ Borel, **2**, 120 and **4**, 32. Borel considers the cases only in which $\psi = ex$, $\phi = ex^2$ or e_2x, but his method is obviously general. The proof given here is however more general and simpler.

of ϕ and ψ may be as large as we like (§ 2.32, Theorem 2). We suppose that $\phi \succ \psi \succ x^{\Delta}$. Then we can define a function

$$f(x) = \Sigma c_n x^n,$$

where every c_n is equal either to a_n or to b_n, in such a way that $f \sim \phi$ for an infinity of values x_ν whose limit is infinity, and $f \sim \psi$ for a similar infinity of values x_ν' *.

Let (η_ν) be a sequence of decreasing positive numbers whose limit is zero. Take a positive number x_0 such that $\phi(x_0) > 1$, $\psi(x_0) > 1$, and a number x_1 greater than x_0. When x_1 is fixed, we can choose n_1 so that

$$\sum_{n_1}^{\infty} a_n x_1^n < \tfrac{1}{3}\eta_1, \quad \sum_{n_1}^{\infty} b_n x_1^n < \tfrac{1}{3}\eta_1,$$

and so, however c_n be selected for different values of n,

$$\sum_{n_1}^{\infty} c_n x_1^n < \sum_{n_1}^{\infty} (a_n + b_n) x_1^n < \tfrac{2}{3}\eta_1.$$

We take $c_n = a_n$ for $0 \leqslant n < n_1$. Then

$$|f(x_1) - \phi(x_1)| < \sum_{n_1}^{\infty} (a_n + c_n) x_1^n < \eta_1,$$

and so, since $\phi(x_1) > 1$,

$$\left| \frac{f(x_1)}{\phi(x_1)} - 1 \right| < \eta_1.$$

Now let x_2 be a number greater than x_1; we can suppose x_2 chosen so that

$$\left(\sum_{0}^{n_1-1} a_n x_2^n\right) \Big/ \psi(x_2) < \tfrac{1}{5}\eta_2, \quad \left(\sum_{0}^{n_1-1} b_n x_2^n\right) \Big/ \psi(x_2) < \tfrac{1}{5}\eta_2.$$

When x_2 is fixed we can choose n_2, greater than n_1, so that

$$\sum_{n_2}^{\infty} a_n x_2^n < \tfrac{1}{5}\eta_2, \quad \sum_{n_2}^{\infty} b_n x_2^n < \tfrac{1}{5}\eta_2.$$

We take $c_n = b_n$ for $n_1 \leqq n < n_2$; and, however c_n be chosen for $n \geqq n_2$, we have

$$\sum_{n_2}^{\infty} c_n x_2^n < \sum_{n_2}^{\infty} (a_n + b_n) x_2^n < \tfrac{2}{5}\eta_2.$$

Also

$$|f(x_2) - \psi(x_2)| < \sum_{0}^{n_1-1} a_n x_2^n + \sum_{0}^{n_1-1} b_n x_2^n + \sum_{n_2}^{\infty} c_n x_2^n + \sum_{n_2}^{\infty} b_n x_2^n$$
$$< \tfrac{2}{5}\eta_2 \psi(x_2) + \tfrac{3}{5}\eta_2 < \eta_2 \psi(x_2),$$

and so

$$\left| \frac{f(x_2)}{\psi(x_2)} - 1 \right| < \eta_2.$$

* By '$f \sim \phi$ for an infinity of values x_ν' we mean of course that $f/\phi \to 1$ when $x \to \infty$ through this particular sequence of values.

It is plain that, by a repetition of this process, we can find a sequence $x_1, x_2, x_3, \ldots$ whose limit is infinity, so that

$$\left|\frac{f(x_3)}{\phi(x_3)}-1\right| < \eta_3, \quad \left|\frac{f(x_4)}{\psi(x_4)}-1\right| < \eta_4, \;\ldots\ldots,$$

and our conclusion is established.

We may remark that not only f itself, but all its derivatives also, are increasing and continuous. It is clear also that, if we were given any number of integral functions $\phi_1, \phi_2, \ldots, \phi_k$, with positive coefficients, we could define f so that $f \sim \phi_s$, as $x \to \infty$ through a suitably chosen sequence of values, for each of the functions ϕ_s.

Another interesting method for the construction of irregularly increasing functions by means of power series will be explained in § 6.34.

4.44. Functions which transcend the logarithmico-exponential scales. We turn our attention now to functions for which the logarithmico-exponential scales are not inapplicable but *insufficient* (§ 4.42). Of the existence of such functions we are already assured; thus a function which assumes the values $e_1(1), e_2(2), \ldots, e_\nu(\nu), \ldots$ for $x = 1, 2, \ldots, \nu, \ldots$ has certainly an increase greater than that of any logarithmico-exponential function.

(i) The series
$$\Sigma \frac{e_\nu(x)}{e_\nu(\nu)},$$
if convergent for all values of x, has a sum $f(x)$ whose increase is plainly greater than that of any $e_\nu(x)$. Suppose that $k-1 \leqq x < k$. Then

$$\frac{e_k(x)}{e_k(k)} < 1, \quad \frac{e_{k+\nu}(x)}{e_{k+\nu}(k+\nu)} < \frac{e_{k+\nu}(k)}{e_{k+\nu}(k+\nu)} < \frac{e_{k+\nu}(k)}{e_{k+\nu}(k+1)} \quad (\nu \geqq 1).$$

But, by the Mean Value Theorem,

$$e_{k+\nu}(k+1) = e_{k+\nu}(k) + e_{k+\nu}(y)\, e_{k+\nu-1}(y) \ldots e_2(y)\, e_1(y),$$

where y is some number between k and $k+1$; and so

$$e_{k+\nu}(k+1) > e_{k+\nu}(k)\, e_{k+\nu-1}(k) \ldots e_1(k).$$

It follows that the terms of the series

$$\sum_{\nu=k}^{\infty} \frac{e_\nu(x)}{e_\nu(\nu)}$$

are less than those of the series

$$1 + \sum_{\nu=1}^{\infty} \frac{1}{e_1(k)\, e_2(k) \ldots e_{k+\nu-1}(k)},$$

which is plainly convergent, so that the original series is convergent. It is obvious that we can in this way construct any number of functions $f(x)$ of the kind required.

(ii) Let $\phi(x)$ be an increasing function such that $\phi(0) > 0$, $\phi \succ x$. We can define an increasing function f, which satisfies the equation

(4.441) $$f_2(x) = f\{f(x)\} = \phi(x),$$

as follows.

Draw the curves $y = x$, $y = \phi(x)$ (Fig. 5). Take Q_0 arbitrarily on OP_0; draw Q_0R_1 parallel to OX and complete the rectangle Q_0Q_1. Join Q_0, Q_1 by any continuous curve inclined everywhere at an acute angle to the axes. On this curve take any point Q; draw QP, QR parallel to the axes, and complete the rectangle QQ'. When Q moves from Q_0 to Q_1, Q' moves from Q_1 to Q_2, say; and as we constructed Q' from Q, so we can construct Q'' from Q'. Proceeding thus, we define a continuous curve $Q_0Q_1Q_2Q_3 \ldots$ corresponding to a continuous and increasing function $f(x)$. Then $f(x)$ satisfies (4.441). For if $y = f(x)$ is the ordinate of Q, it is clear that $f_2(x)$ is the ordinate of Q', which is equal to $\phi(x)$, the ordinate of P.

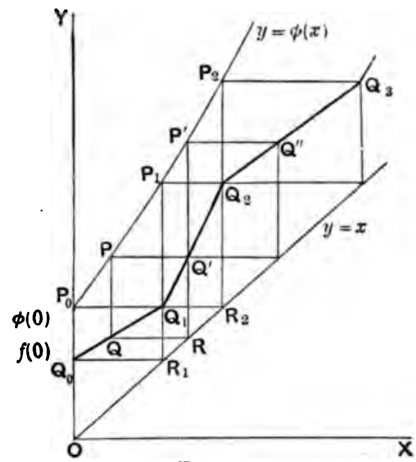

FIG. 5

Let us write $f(x) = f_1(x)$ and $f\{f_n(x)\} = f_{n+1}(x)$, so that Q_n is the point $f_n(0)$, $f_{n+1}(0)$. Further, let us suppose that ψ is the function inverse to ϕ, that $\psi(x) = \psi_1(x)$, $\psi\{\psi(x)\} = \psi_2(x)$, ...; and that the equation of Q_0Q_1 is $\theta(x, y) = 0$. Then it is easy to see that the equations of $Q_{2n}Q_{2n+1}$ and of $Q_{2n+1}Q_{2n+2}$ are respectively

$$\theta\{\psi_n(x), \psi_n(y)\} = 0, \quad \theta\{\psi_{n+1}(y), \psi_n(x)\} = 0.$$

Suppose for example that $\phi(x) = e^x$, $OQ_0 = \frac{1}{2}$, and that Q_0Q_1 is the straight line $y = \frac{1}{2} + x$. Then the equations of $Q_{2n}Q_{2n+1}$ and of $Q_{2n+1}Q_{2n+2}$ are

$$l_n y = \tfrac{1}{2} + l_n x, \quad l_n x = \tfrac{1}{2} + l_{n+1} y,$$

or $$y = e_{n-1}\{\sqrt{e}\,(l_{n-1}x)\} = e_{n-2}\{(l_{n-2}x)^{\sqrt{e}}\} = \lambda_n(x),$$

and $$y = e_n\{(l_{n-1}x)/\sqrt{e}\} = e_{n-1}\{(l_{n-2}x)^{1/\sqrt{e}}\} = \mu_n(x),$$

say. Now (§ 4.1)

$$x^{\gamma} \prec \lambda_3 \prec \ldots \prec \lambda_n \prec \ldots \prec \mu_n \prec \ldots \prec \mu_3 \prec e^{x^{\gamma}},$$

and a function f, such that $\lambda_n \prec f \prec \mu_n$ for all values of n, transcends the logarithmico-exponential scales. Our function f is plainly an example*.

It is easily verified that $\lambda_n \lambda_n x \prec e^x$ and $\mu_n \mu_n x \succ e^x$ for all values of n. Hence it is clear *a priori* that any increasing solution of (1) must satisfy $\lambda_n \prec f \prec \mu_n$ for all values of n.

* For fuller details see Hardy, 9.

This graphical method may also be employed to define functions whose increase is slower than that of any logarithm or more rapid than that of any exponential. It can be employed, for example, to solve the equation

$$\phi(2^x)=2\phi(x);$$

and it is easily proved that the increase of a function such that $\phi(2^x)\asymp\phi(x)$ is slower than that of any logarithm.

4.5. The importance of the logarithmico-exponential scales. We have seen that it is possible, in a variety of ways, to construct functions whose increase cannot be measured by any L-function. It is none the less true that no one yet has succeeded in defining a mode of increase which is genuinely independent of all logarithmico-exponential modes. Our irregularly increasing functions oscillate, according to a logarithmico-exponential law of oscillation, between two logarithmico-exponential functions; and the function of § 4.44 (ii) was constructed expressly to fill a gap between two logarithmico-exponential scales. No function has yet presented itself in analysis the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to say, in logarithmico-exponential terms.

It would be natural to expect that the arithmetical functions which occur in the theory of numbers might give rise to genuinely new modes of increase; but, so far as analysis has gone, the evidence is the other way. See § 6.26.

V

DIFFERENTIATION AND INTEGRATION

5.1. Integration. It is important to know when relations of the types $f(x)\succ\phi(x)$, etc., can be differentiated or integrated*. For brevity we denote

$$\int_a^x f(t)\,dt,\quad \int_a^x \phi(t)\,dt$$

(where a is a constant) by $F(x)$ and $\Phi(x)$.

It may be well to repeat that f and ϕ are supposed to be positive, continuous, and monotonic (at any rate for $x>x_0$), unless the contrary is stated or clearly implied. Some of our conclusions are valid under more general conditions; but the case thus defined, and the corresponding cases in which f or ϕ or both of them are negative, are the cases of most importance.

* The problem was first considered generally by du Bois-Reymond, **1**, **2**, **4**.

Lemma. *If* $\Phi \succ 1$, *and* $f > H\phi$ *for* $x > x_0 \geqq a$, *then* x_1 *can be found so that* $F > (H - \delta)\,\Phi$ *for* $x > x_1$: *similarly* $f < h\phi$ *for* $x > x_0$ *involves* $F < (h + \delta)\,\Phi$ *for* $x > x_1$.

For, if $f > H\phi$ for $x > x_0$, we have

$$F = \int_a^x f\,dt > \int_a^{x_0} f\,dt + H\int_{x_0}^x \phi\,dt = H\Phi + \int_a^{x_0} f\,dt - H\int_a^{x_0} \phi\,dt.$$

Since $\Phi \succ 1$, we can choose x_1 so that

$$\frac{1}{\Phi}\left(\int_a^{x_0} f\,dt + H\int_a^{x_0} \phi\,dt\right) < \delta$$

for $x > x_1$; and the first part of the lemma follows. The second part may be proved similarly. From the lemma we can at once deduce

Theorem 19. *If either* $F \succ 1$ *or* $\Phi \succ 1$, *then any one of the relations*

$$f \succ \phi,\ f \prec \phi,\ f \asymp \phi,\ f \succeq\!\!\preceq \phi,\ f \sim \phi$$

involves the corresponding one of the relations

$$F \succ \Phi,\ F \prec \Phi,\ F \asymp \Phi,\ F \succeq\!\!\preceq \Phi,\ F \sim \Phi.$$

To this we may add

Theorem 20. *If both* $\int^\infty f\,dt$, $\int^\infty \phi\,dt$ *are convergent, then*

$$f \succ \phi,\ f \prec \phi,\ f \asymp \phi,\ f \succeq\!\!\preceq \phi,\ f \sim \phi$$

involve corresponding relations between

$$F_1 = \int_x^\infty f\,dt, \quad \Phi_1 = \int_x^\infty \phi\,dt.$$

The proof we may leave to the reader.

5.21. Differentiation. From Theorems 19 and 20 we deduce

Theorem 21. *If* $f \succ 1$ *or* $\phi \succ 1$, *or* $f \prec 1$ *and* $\phi \prec 1$, *and if some one of the relations* $\succ$, $\prec$, $\asymp$, $\succeq\!\!\preceq$, $\sim$ *must hold between* f' *and* ϕ', *then* $f \succ \phi$ *involves* $f' \succ \phi'$; *and there are corresponding results for the other relations.*

In interpreting this theorem regard must be paid to the conventions laid down in § 1.4. Thus if $f \succ \phi \succ 1$, f' and ϕ' are positive, and $f' \succ \phi'$. But if $f \succ 1 \succ \phi$, ϕ is a decreasing function and $\phi' < 0$. In this case $f' \succ -\phi'$, a relation which we have agreed to denote by $f' \succ \phi'$. If $1 \succ f \succ \phi$, both f' and ϕ' are negative: the relation $-f' \prec -\phi'$ would involve

$$-\int_x^\infty f'\,dt \prec -\int_x^\infty \phi'\,dt$$

or $f \prec \phi$, and is therefore impossible; and similarly $-f' \asymp -\phi'$ is impossible. We must therefore have $-f' \succ -\phi'$, a relation which we have agreed to denote also by $f' \succ \phi'$. The case in which $f \asymp 1$, $\phi \preccurlyeq 1$, is exceptional; any one of the relations $f' \succ \phi'$, etc. may hold in this case. Thus if $f = 1 + e^{-x}$, $\phi = 1/x$, we have $f \succ \phi$, $f' \prec \phi'$. The fact is that in this case f, regarded as the integral of f', is dominated by the constant of integration.

It is to be observed that the assumption that one of the relations holds between f' and ϕ' is essential. We cannot *infer* that one of them holds; we cannot even infer that f' or ϕ' is a steadily increasing or decreasing function. Thus if $f = e^x$, $\phi = e^x + \sin e^x$, we have $f' = e^x$ and $\phi' = e^x(1 + \cos e^x)$. Here f and ϕ increase steadily and $f' \sim f \sim \phi$; but ϕ' does not tend to infinity, and in fact vanishes for an infinity of values of x. Again if

$$\phi = e^x(\sqrt{2} + \sin x) + \tfrac{1}{2}x^2,$$

we have

$$\phi' = e^x(\sqrt{2} + \sin x + \cos x) + x$$

and $\phi \asymp e^x$, while ϕ' oscillates between the orders of e^x and x. It is possible, though less easy, to obtain examples of this character in which ϕ' also is monotonic.

5.22. Differentiation of *L*-functions. If f and ϕ are L-functions, so are f' and ϕ', and one of the relations $f' \succ \phi'$, $f' \succcurlyeq\!\preccurlyeq \phi'$, $f' \prec \phi'$ certainly holds (§ 3.2, Theorem 13). Thus in this case *both differentiation and integration are always legitimate* except when* $f \asymp 1$, $\phi \preccurlyeq 1$, *or* $f \preccurlyeq 1$, $\phi \asymp 1$.

In what follows we shall suppose that all the functions concerned are L-functions, or at any rate resemble L-functions in so far that one of the relations $f \succ \phi$, $f \succcurlyeq\!\preccurlyeq \phi$, $f \prec \phi$ is bound to hold between any pair of functions, and that differentiation and integration are permissible†.

Theorem 22. *If f is an increasing function, and $f' \succ f$, then $f \succ e^{\Delta x}$. If $f' \prec f$, then $f \prec e^{\delta x}$. Similarly, if f is a decreasing function, $f' \succ f$ and $f' \prec f$ involve $f \prec e^{-\Delta x}$ and $f \succ e^{-\delta x}$ respectively. If $f' \succcurlyeq\!\preccurlyeq f$, then we can find a number μ such that $f = e^{\mu x} f_1$, where $e^{-\delta x} \prec f_1 \prec e^{\delta x}$.*

The proofs of these propositions are almost obvious. Thus if f is an increasing function, and $f' \succ f$, we have

$$f'/f \succ 1, \quad \log f \succ x,$$

and so $\log f > \Delta x$ for $x > x_0$, *i.e.* $f > e^{\Delta x}$, or, what is the same thing, $f \succ e^{\Delta x}$. The last clause of the theorem follows at once from § 3.5.

Theorem 23. *More generally, if v is any increasing function, $f'/f \succ v'/v$ involves $f \succ v^{\Delta}$ or $f \prec v^{-\Delta}$, according as f is an increasing or a decreasing function; and $f'/f \prec v'/v$ involves $f \prec v^{\delta}$ or $f \succ v^{-\delta}$. If $f'/f \succcurlyeq\!\preccurlyeq v'/v$, we can find a number μ such that $f = v^{\mu} f_1$, where $v^{-\delta} \prec f_1 \prec v^{\delta}$.*

* A tacit assumption to this effect underlies much of du Bois-Reymond's work.

† The results which follow are all in substance due to du Bois-Reymond.

When f is an increasing function, which tends to infinity with x, we shall call f'/f the *type* t of f*: it being understood that t may be replaced by any simpler function τ such that $t \asymp \tau$. The type of a *decreasing* function f we define to be the same as that of the increasing function $1/f$. The following table shows the types of some standard functions:

Function ...	llx	lx	x^a	e^x	e^{ax^β}	e_2x	e_3x	...
Type ...	$\dfrac{1}{x\,lx\,llx}$	$\dfrac{1}{x\,lx}$	$\dfrac{1}{x}$	1	$x^{\beta-1}$	ex	$e_2x\,ex$	...

If $f \succ \phi$, then $f'/f \succcurlyeq \phi'/\phi$. By making the increase of f large enough we can make the increase of $t = f'/f$ as large as we please. The reader will find it instructive to write out formal proofs of these propositions, and also of the following.

1. As the increase of f becomes smaller and smaller, f'/f tends to zero more and more rapidly, but, so long as $f \to \infty$, we cannot have

$$\frac{f'}{f} \prec \phi(x), \quad \int^{\infty} \phi\, dx \text{ convergent.}$$

If on the other hand the last integral is divergent, we can always find f so that $f \succ 1$, $f'/f \prec \phi$.

2. Although we can find f so that f'/f shall have an increase larger than that of any given function of x, we cannot have

$$\frac{f'}{f} \succ \phi(f), \quad \int^{\infty} \frac{dx}{x\phi(x)} \text{ convergent.}$$

If on the other hand the last integral is divergent, we can always find f so that $f'/f \succ \phi(f)$.

Thus we cannot find a function f which tends to infinity so slowly that $f'/f \prec 1/x^a$ $(a > 1)$. But we can find f so that $f'/f \prec 1/x lx llx$ (*e.g.* $f = l_3x$). We cannot find f so that $f'/f \succ f^a$ or $f' \succ f^{1+a}$ $(a > 0)$. But we can find f so that $f'/f \succ lf$ (*e.g.* $f = e_3x$).

3. If $f \succ e_k x$ for all values of k, f'/f satisfies the same condition, and

$$f' \succ f\,lf\,l_2f \dots l_kf\dagger.$$

There are of course corresponding theorems about functions of a positive variable x which tends to zero.

5.23. Successive differentiation. du Bois-Reymond‡ has given the following general theorem, which enables us to write down the increase of any derivative of any logarithmico-exponential function. We write t for f'/f, as in the last section.

* du Bois-Reymond (**1**, **2**) calls f/f' the type; the notation here adopted seems slightly more convenient.

† In this case f cannot be an L-function (§ 3.4, Theorem 14). It is however supposed to possess the properties stated at the beginning of this section.

‡ du Bois-Reymond, **2**.

Theorem 24. (i) *If* $t \succ 1/x$ (*so that* $f \succ x^{\Delta}$ *or* $f \prec x^{-\Delta}$) *then*

$$f \asymp f'/t \asymp f''/t^2 \asymp f'''/t^3 \ldots \asymp f^{(n)}/t^n \ldots.$$

(ii) *If* $t \prec 1/x$ (*so that* $1 \prec f \prec x^{\delta}$ *or* $x^{-\delta} \prec f \prec 1$) *then*

$$f \asymp f'/t \asymp xf''/t \asymp x^2 f'''/t \ldots \asymp x^{n-1} f^{(n)}/t \ldots.$$

(iii) *If* $t \asymp 1/x$ (*so that* $f = x^{\mu} f_1$, *where* $x^{-\delta} \prec f_1 \prec x^{\delta}$), *then if* μ *is not integral either set of formulae is valid. If* μ *is integral then*

$$f \asymp xf' \asymp x^2 f'' \ldots \asymp x^{\mu} f^{(\mu)} \asymp x^{\mu} f^{(\mu+1)}/t_1 \asymp x^{\mu+1} f^{(\mu+2)}/t_1 \ldots,$$

where t_1 *is the type of* f_1, *unless* $f_1 \asymp 1$.

(i) If $t \succ 1/x$, $1/t \prec x$ and so $t'/t^2 \prec 1$; hence $t'/t \prec t = f'/f$ or

$$ft' \prec f't.$$

Differentiating the relation $f' \asymp ft$, and using the relation just established, we obtain

$$f'' \asymp f't + ft' \asymp f't.$$

Thus the type of f' is the same as that of f. The argument may therefore be repeated, and the first part of the theorem follows.

(ii) If $t \prec 1/x$, $xf' \prec f$ and so

$$xf'' + f' \prec f',$$

which cannot be true unless $xf'' \asymp f'$. Differentiating again we infer

$$xf''' + 2f'' \prec f'',$$

whence $xf''' \asymp f''$; and so on generally*. Thus the second part follows.

(iii) If $t \asymp 1/x$, $f = x^{\mu} f_1$ and t_1, the type of f_1, satisfies $t_1 \prec 1/x$. Then

$$f' = \mu x^{\mu-1} f_1 + x^{\mu} f_1' \asymp x^{\mu-1} f_1 (\mu + xt_1) \asymp x^{\mu-1} f_1.$$

Similarly $f'' \asymp x^{\mu-2} f_1$, and so on. We can proceed indefinitely in this way unless μ is integral: in this case $f^{(\mu)} \asymp f_1$, and from this point we proceed as in case (ii).

If μ is an integer n, and $f_1 \asymp 1$, then $f^{(n)} \asymp 1$, but the theorem fails for the higher derivatives. In this case $f = Ax^n + o(x^n) = Ax^n + \phi$, say, and we must begin our analysis again with ϕ in place of f.

Examples. (i) If $f = e^{\sqrt{x}}$, then $t = x^{-\frac{1}{2}} \succ 1/x$, and $f^{(n)} \asymp x^{-\frac{1}{2}n} e^{\sqrt{x}}$. If $f = e^{(\log x)^2}$, then $t = (\log x)/x \succ 1/x$, and $f^{(n)} \asymp e^{(\log x)^2} (\log x)^n / x^n$.

(ii) If $f = (\log x)^m$, then $t = 1/(x \log x) \prec 1/x$, and

$$f^{(n)} \asymp tx^{-(n-1)} f \asymp (\log x)^{m-1}/x^n.$$

(iii) If $f = x^2 llx$, $t \asymp 1/x$. Here

$$f' \asymp x\,llx,\ f'' \asymp llx,\ f''' \asymp 1/x\,lx,\ f'''' \asymp 1/x^2 lx, \ldots.$$

* More precisely $xf'' \sim -f'$, $xf''' \sim -2f''$, and so on.

(iv) The results of the theorem, in the first two cases, can be stated more precisely as follows. If $t \succ 1/x$, then

$$f^{(n)} \sim (f'/f)^n f.$$

If $t \prec 1/x$, then

$$f^{(n)} \sim (-1)^{n-1}(n-1)!\, f'/x^{n-1}.$$

If f is a positive increasing function, and $t \succ 1/x$, then all the derivatives are ultimately positive. If $t \prec 1/x$, they are alternately positive and negative.

5.3. Further theorems on integration. It is possible to give a finite system of rules which enable us to determine the asymptotic behaviour of the integral of any L-function. The results are naturally not essentially different from those of § 5.23. We write

$$F(x) = \int_a^x f(t)\,dt, \quad F(x) = \int_x^\infty f(t)\,dt$$

according as the latter integral is divergent or convergent.

Theorem 25. *If $f \succ x^\Delta$ or $f \prec x^{-\Delta}$, then*

$$F \sim f^2/f'.$$

If $f = x^a f_1$, where $x^{-\delta} \prec f_1 \prec x^\delta$, then

$$F \sim \frac{x^{a+1}}{a+1} f_1,$$

unless $a = -1$, in which case further rules are required.

(1) If $f \succ x^\Delta$, the integral up to infinity is divergent, and

$$F = \int_a^x f\,dt = \int_a^x f' \frac{f}{f'}\,dt = \frac{\{f(x)\}^2}{f'(x)} - \frac{\{f(a)\}^2}{f'(a)} - \int_a^x f \frac{d}{dt}\left(\frac{f}{f'}\right) dt.$$

Now $\log f \succ \log x$, and so

$$\frac{f'}{f} \succ \frac{1}{x}, \quad x \succ \frac{f}{f'}, \quad 1 \succ \frac{d}{dx}\left(\frac{f}{f'}\right), \quad \int_a^x f\,dt \succ \int_a^x f \frac{d}{dt}\left(\frac{f}{f'}\right) dt,$$

so that $F \sim f^2/f'$. The case in which $f \prec x^{-\Delta}$ may be disposed of similarly.

(2) If $f = x^a f_1$, where $a > -1$, the integral up to infinity is again divergent; and

$$F = \int_a^x t^a f_1\,dt = \frac{x^{a+1}}{a+1} f_1(x) - \frac{a^{a+1}}{a+1} f_1(a) - \frac{1}{a+1}\int_a^x t^{a+1} f_1'\,dt.$$

But

$$\log f_1 \prec \log x, \quad \frac{f_1'}{f_1} \prec \frac{1}{x}, \quad x^{a+1} f_1' \prec x^a f_1, \quad \int_a^x t^{a+1} f_1'\,dt \prec \int_a^x t^a f_1\,dt;$$

whence the result. The case in which $a < -1$ is not essentially different.

When $a = -1$, further analysis is required, which will be found in the author's paper quoted on p. 21.

Another interesting problem is that of the behaviour of F when $f = \phi e^{i\psi}$, ϕ and ψ being L-functions. Let us suppose that $\psi \succ 1$ and $\phi \succ \psi'$, so that the integral does not converge up to infinity*. Then the problem is solved by

* If $\psi \preccurlyeq 1$, $e^{i\psi}$ tends to a limit, and the oscillating factor introduces no new feature. If $\phi \prec \psi'$, the integral up to infinity is convergent.

Theorem 26. *If $\psi \succ 1$, $\phi \succ \psi'$, then F is asymptotically equivalent to*

$$\Phi e^{i\psi},\quad \frac{\Phi e^{i\psi}}{1+Ai},\quad \frac{\phi}{i\psi'}e^{i\psi},$$

where

$$\Phi \sim \int^x \phi\, dt,$$

according as $\psi \prec l\Phi$, $\psi \sim Al\Phi$, or $\psi \succ l\Phi$.

The details of the proof will be found in a note by the author*.

5.4. Some 'Tauberian' theorems. We pointed out in § 5.21 that inferences from the order of magnitude of a function to that of its derivative are essentially more difficult than inferences in the opposite direction, and that special conditions are always required in order that any such inference should be possible. The hypothesis of §§ 5.22—5.23, that the functions concerned are L-functions, is of course an assumption of a very drastic kind. In this section we abandon this hypothesis, and prove some theorems of a more general and much more subtle type. These theorems belong to the class which Mr Littlewood and the author have called 'Tauberian'.

Theorem 27†. *If $xf(x)$ is continuous and increasing for $x > a$, and*

$$F(x) = \int_a^x f\,dt \sim Ax^m \quad (m > 0),$$

then

$$f(x) \sim mAx^{m-1}.$$

The converse inference would be an immediate corollary of Theorem 19.

We may suppose $A = 1$, so that $F = x^m + o(x^m)$. Hence, if η is positive and less than 1, we have

$$F(x+\eta x) - F(x) = \int_x^{x+\eta x} f\,dt = \{(1+\eta)^m - 1\}\,x^m + o(x^m)$$
$$= m\eta x^m + O(\eta^2 x^m) + o(x^m),$$

where $O(\eta^2 x^m)$ is a function whose modulus is less than a constant multiple of $\eta^2 x^m$ for all values of x and η in question. But

$$\int_x^{x+\eta x} f\,dt \geqq \frac{\eta x f(x)}{1+\eta},$$

since tf increases throughout the range of integration. Combining this inequality with the preceding equation, and dividing by $\eta x^m/(1+\eta)$, we obtain

$$\frac{f(x)}{x^{m-1}} \leqq m(1+\eta) + H\eta + o(1),$$

* Hardy, 3 (6). † Landau, 2, 218 and 3, 116.

where H is independent of m and of η. If now we make $x \to \infty$, we find

$$\overline{\lim_{x\to\infty}} \frac{f(x)}{x^{m-1}} \leqq m(1+\eta) + H\eta;$$

and this involves

$$\overline{\lim_{x\to\infty}} \frac{f(x)}{x^{m-1}} \leqq m,$$

since η may be as small as we please. Arguing in the same way with the interval $(x - \eta x, x)$, we obtain

$$\underline{\lim_{x\to\infty}} \frac{f(x)}{x^{m-1}} \geqq m;$$

and these two inequalities embody the result of the theorem.

Theorem 28. *If* $(1-x)f'(x)$ *is continuous and increasing for* $0 < x < 1$, *and*

(5.41) $$f(x) \sim \frac{A}{(1-x)^m} \quad (m > 0)$$

when $x \to 1$, *then*

(5.42) $$f'(x) \sim \frac{mA}{(1-x)^{m+1}}.$$

We have only to write

$$x = \frac{1}{1-y}, \quad f(x) = g(y)$$

in Theorem 27, and then replace y by x.

Theorem 29*. *If* $f(x) = \Sigma a_n x^n$ *is a power series with positive coefficients, convergent for* $0 \leqq x < 1$, *then* (5.41) *involves* (5.42).

Let $a_0 + a_1 + \dots + a_n = A_n$, so that

$$g(x) = \sum_0^\infty A_n x^n = \frac{f(x)}{1-x} \sim \frac{A}{(1-x)^{m+1}}.$$

Then $(1-x)g'(x) = A_1 + (2A_2 - A_1)x + (3A_3 - 2A_2)x^2 + \dots$ increases steadily, since the coefficients are positive. Hence, by Theorem 28,

$$g'(x) \sim \frac{(m+1)A}{(1-x)^{m+2}},$$

$$f'(x) = (1-x)g'(x) - g(x) \sim \frac{mA}{(1-x)^{m+1}}.$$

Theorem 30. *If* $f(x)$ *possesses a second derivative* $f''(x)$, *and* $f = O(x^\alpha)$, $f'' = O(x^\beta)$, *where* $\beta > -1$, *when* $x \to \infty$, *then*

$$f' = O\{x^{\frac{1}{2}(\alpha+\beta)}\}.$$

If $\alpha \geqq \beta + 2$ the result is trivial, since $f' = O(x^{\beta+1})$, by Theorem 19,

* Hardy and Littlewood, 2.

and $\beta+1 \leqq \frac{1}{2}(\alpha+\beta)$. We may therefore suppose that $\alpha<\beta+2$. If $0<\eta<1$ we have, by Taylor's Theorem,

$$f(x+\eta x)-f(x)=\eta x f'(x)+\tfrac{1}{2}\eta^2 x^2 f''(x+\theta\eta x),$$

where $0<\theta<1$; and so

$$|f'(x)| \leqq \frac{|f(x+\eta x)|+|f(x)|}{\eta x}+\tfrac{1}{2}\eta x\,|f''(x+\theta\eta x)|<H\left(\frac{x^{\alpha-1}}{\eta}+\eta x^{\beta+1}\right),$$

where H is independent of x and η. We may take $\eta^2=x^{\alpha-\beta-2}$, since this is certainly less than 1 when x is large; and then

$$|f'(x)|<2Hx^{\frac{1}{2}(\alpha+\beta)},$$

which proves the theorem.

The theorem is not true if $\beta \leqq -1$: consider, for example, the case $f=x+\log x$. It is one of a system of theorems important in the theory of infinite series*.

5.5. Functions of an integral variable. There are theorems for functions of an integral variable n, corresponding to those of §§ 5.1—5.4, but involving sums

$$A_n=a_1+a_2+\ldots+a_n$$

instead of integrals, and differences

$$\Delta a_n=a_n-a_{n+1}$$

instead of differential coefficients. The reader will be able to formulate and prove for himself the theorems which correspond to those of the preceding paragraphs. Thus

> '$a_n \succ b_n$, $a_n \prec b_n$, $a_n \asymp b_n$, $a_n \succcurlyeq\!\!\preccurlyeq b_n$, $a_n \sim b_n$ *involve the corresponding equations for* A_n *and* B_n, *if one at least of* A_n *and* B_n *tends to infinity with* n';

and so on†.

5.6. Further developments of the Infinitärcalcül. The functions $f(x+a)$, $f(ax)$, etc. It is often necessary to obtain approximations to such functions as

$$\frac{f(x+a)}{f(x)},\quad \frac{f(ax)}{f(x)},\quad f(x+a)-f(x),$$

where a is itself a function of x‡. We shall assume that all the functions which occur are L-functions, or at any rate that the theorems of §§ 5.2—5.3 may be applied to them as if they were.

* See Hardy and Littlewood, **1**, **2**.

† This is a well known theorem of Cauchy and Stolz: see Bromwich, **1**, 377; Knopp, **1**, 72.

‡ du Bois-Reymond, **4**. The substance of the theorems which follow is in the main due to du Bois-Reymond; but his presentation of them is inconclusive.

Theorem 31. *If* $a \prec f/f'$ *then*

$$\frac{f(x+a)}{f(x)} \sim 1.$$

We may suppose that $f \succ 1$ and $a > 0$. If, in the first place, $t = f'/f \preccurlyeq 1$, we have

$$\frac{f(x+a)}{f(x)} = e^{lf(x+a)-lf(x)} = e\left\{a\frac{f'(x+\alpha)}{f(x+\alpha)}\right\} = e\{a\,t(x+\alpha)\},$$

and $t(x+\alpha) < Kt(x)$, so that $a\,t(x+\alpha) \rightarrow 0$, which proves the theorem.

If $t \succ 1$, and $T = 1/t$, we have

$$a\,t(x+\alpha) = a\,t(x)\frac{T(x)}{T(x+\alpha)} = a\,t(x)\Big/\left\{1+\alpha\frac{T'(x+\alpha_1)}{T(x)}\right\},$$

where $0 < \alpha_1 < \alpha < a$. But $at \prec 1$, $a/T \prec 1$, and $T' \prec 1$ (since $T \prec 1$). Hence $a\,t(x+\alpha) \rightarrow 0$, which again proves the theorem.

In particular the conditions are satisfied if (i) $x^{-\Delta} \prec f \prec x^{\Delta}$ and $a \prec x$ or (ii) $e^{-\Delta x} \prec f \prec e^{\Delta x}$ and $a \prec 1$.

Theorem 32. *If* $la \prec f/xf'$ *then*

$$\frac{f(ax)}{f(x)} \sim 1.$$

This is a corollary of Theorem 31: we have only to write $lx = y$, $la = b$, and $f(x) = \phi(y)$.

In particular the conditions are satisfied if $(lx)^{-\Delta} \prec f \prec (lx)^{\Delta}$ and $x^{-\delta} \prec a \prec x^{\delta}$ or if $x^{-\Delta} \prec f \prec x^{\Delta}$ and $a \sim 1$.

We add some further results.

(1) *If* $a \prec 1/f'$ *then* $f(x+a) - f(x) \prec 1$.

(2) *If* $a \prec f'/f''$ *then* $f(x+a) - f(x) \sim af'(x)$.

These results follow from Theorem 31 and the formula

$$f(x+a) - f(x) = af'(x)\frac{f'(x+\alpha)}{f'(x)} \quad (0 < \alpha < a).$$

The second result is true in particular if $1 \prec f \prec x^{\delta}$ and $a \prec x$, or if $f \succ x^{\Delta}$ and $a \prec f/f'$; the forms of the conditions to be imposed on a may be deduced from Theorem 24.

(3) *If* $e^{-\Delta\sqrt{(lx)}} \prec f \prec e^{\Delta\sqrt{(lx)}}$, *then*

$$\frac{f\{xf(x)\}}{f(x)} \asymp 1, \quad e\left\{\frac{xlf(x)f'(x)}{f(x)}\right\} \asymp 1;$$

and the limits of the two functions are the same: and if $e^{-\delta\sqrt{(lx)}} \prec f \prec e^{\delta\sqrt{(lx)}}$ *this limit is unity.*

Suppose that $f \succ 1$, and let $f(x)=\phi(lx)$, $a=f(x)$. Then

$$\frac{f(ax)}{f(x)}=e^{l\phi(lx+la)-l\phi(lx)}=e^{la\phi'(lx+la_1)/\phi(lx+la_1)},$$

where $1<a_1<a$. The exponent is

$$l\phi(lx+la_1)\frac{\phi'(lx+la_1)}{\phi(lx+la_1)}\frac{l\phi(lx)}{l\phi(lx+la_1)}.$$

Now $a=f(x)\prec x^\delta$ and therefore $la_1 \preccurlyeq la \prec lx$, and so, by Theorem 31,

$$l\phi(lx+la_1)\sim l\phi(lx)$$

if $l\phi \prec x^\Delta$ or if $f\prec e^{(lx)^\Delta}$, which is certainly the case. Hence the exponent is asymptotically equivalent to

$$l\phi(u)\,\phi'(u)/\phi(u),$$

where $u=lx+la_1$. And $l\phi(\phi'/\phi)\preccurlyeq 1$ if $(l\phi)^2\preccurlyeq u$, *i.e.* if $\phi\preccurlyeq e^{\Delta\sqrt{u}}$ or $f\preccurlyeq e^{\Delta\sqrt{(lx)}}$. In this case $f(ax)\asymp f(x)$; and it is easy to see that if $f\preccurlyeq e^{\delta\sqrt{(lx)}}$ the symbol $\asymp$ may be replaced by $\sim$.

(4) *If* $f(x)=x\phi(x)$, *and* $e^{-\delta\sqrt{(lx)}}\prec\phi\prec e^{\delta\sqrt{(lx)}}$, *then*

$$f_2(x)\equiv ff(x)\sim x\phi^2,\ \ldots,\ f_n\sim x\phi^n,\ \ldots.$$

5.7. Approximate solution of equations. We may say that

$$y=\psi(x,u)$$

is an 'approximate form' of y if ψ is a known function and u an unknown function whose increase is subject to known limitations. Thus

$$e^{x^u}\ (u\sim 1),\quad e^{(1+u)x}\ (u\prec 1),\quad x^{1+u}e^x\ (u\prec 1)$$

are approximate forms of $y=xe^x/lx$, and represent the increase of y with increasing accuracies. Another example of an approximation is given by the formula

$$\frac{f(x+a)}{f(x)}=e\left\{u\frac{f'(x)}{f(x)}\right\}\quad (u\sim a),$$

valid if $a\prec f/f'\prec 1$.

It is often important to obtain an asymptotic solution of an equation $f(x,y)=0$, *i.e.* to find a function whose increase gives an approximation to that of y. No very general methods of procedure can be given, but the kind of methods which may be pursued are worth illustrating by a few examples.

Suppose that the equation is

$$x=y\kappa(y), \tag{5.71}$$

where $y^{-\delta}\prec\kappa\prec y^\delta$. If the increase of κ is so slow that $\kappa\{y\kappa(y)\}\asymp\kappa(y)$, it is clear that

$$y\asymp x/\kappa(y)\asymp x/\kappa(x):$$

and if the increase of κ is slow enough we may have $y\sim x/\kappa(x)$.

The conditions

$$e^{-\Delta\sqrt{(ly)}}\prec\kappa(y)\prec e^{\Delta\sqrt{(ly)}},\quad e^{-\delta\sqrt{(ly)}}\prec\kappa(y)\prec e^{\delta\sqrt{(ly)}}$$

are, by (3) of § 5.6, enough to ensure the truth of these hypotheses; and then $y=ux/\kappa(x)$, where $u\asymp 1$ (or $u\sim 1$), is an approximate solution of our equation.

du Bois-Reymond has proved that more elaborate approximations, such as

$$y = \frac{ux}{\kappa\,(x/\kappa)},$$

have a wider range of validity. The more general equation $x = y^m \kappa\,(y)$ can clearly be reduced to the form considered above by writing x^m for x and κ^m for κ.

In general, if $x = \phi\,(y)$, the more rapid the increase of ϕ the more precisely can we determine the increase of y as a function of x. Thus if $x = ye^y$ we have $lx = y + ly$ and

$$y = lx - ly = lx\,(1 - u),$$

where $u \sim ly/lx \sim llx/lx$. This is a solution of a much more precise kind than those considered above.

The reader will find it instructive to verify the following examples.

(1) If $x = ye^{(ly)^{\frac{3}{8}}}$, then $y \sim xe^{-(lx)^{\frac{3}{8}}}$.

(2) If $x = ye^{(ly)^{\frac{5}{8}}}$, then

$$y \sim xe\{-(lx)^{\frac{5}{8}} + \tfrac{5}{8}(lx)^{\frac{1}{4}}\}.$$

(3) If $x = y^m (ly)^{m_1} (l_2 y)^{m_2} \dots (l_r y)^{m_r}$, then

$$y \sim m^{m_1/m} x^{1/m} (lx)^{-m_1/m} \dots (l_r x)^{-m_r/m}.$$

(4) If $x = y/ly$, then

$$y = x\left(lx + l_2 x + \frac{l_2 x}{lx}\right) + O\left\{x\,\frac{(l_2 x)^2}{(lx)^2}\right\}.$$

The last example is of interest in the theory of primes.

VI

APPLICATIONS

6.1. In this chapter we give a brief sketch of certain regions of analysis in which the ideas of which we have given an account are of dominating importance.

6.21. Convergence and divergence of series and integrals. The logarithmic tests. A series Σu_n of positive terms is convergent if

$$u_n \preccurlyeq (n\,ln \dots l_{k-1}n)^{-1} (l_k n)^{-1-\alpha},$$

where $\alpha > 0$, and divergent if

$$u_n \succcurlyeq (n\,ln \dots l_k n)^{-1}.$$

Here $k \geqq 0$ and $l_0 n = n$.

An integral $\int^{\infty} f(x)\,dx$, with positive integrand, is convergent if

$$f \preccurlyeq (x\,lx \dots l_{k-1}x)^{-1} (l_k x)^{-1-\alpha},$$

where $\alpha > 0$, and divergent if

$$f \succcurlyeq (xlx \dots l_k x)^{-1}.$$

Similarly the integral $\int_0 f(x)\,dx$ is convergent if

$$f \preccurlyeq (1/x)\{l(1/x) \dots l_{k-1}(1/x)\}^{-1}\{l_k(1/x)\}^{-1-\alpha},$$

where $\alpha > 0$, and divergent if

$$f \succcurlyeq (1/x)\{l(1/x) \dots l_k(1/x)\}^{-1}.$$

These results are classical. The first general statement of the 'logarithmic criteria', so far as series are concerned, appears to have been made by De Morgan, **1**, 325. The essentials, however, appear in a posthumous memoir of Abel (**2**) also first published in 1839: see also Abel, **1**. The case of $k=1$ had been dealt with by Cauchy, **2**. Bertrand (**1**) arrived at De Morgan's results independently, and the criteria are very commonly attributed to him. The first general and explicit statement of the criteria for integrals seems to be due to Bonnet, **1**.

For further information concerning the logarithmic tests, and the corresponding 'ratio-tests' for the convergence of series, see Bromwich, **1**, 29; du Bois-Reymond, **3**; Goursat, **1** (1), 403; Hardy, **1**, 374; Knopp, **1**, 117; Pringsheim, **1** (310), **2** (77), **3**; Riemann, **1**.

6.22. Theorems analogous to du Bois-Reymond's Theorem. We should mention also certain theorems of a negative character, analogous to du Bois-Reymond's theorem of § 2.1.

Given any divergent series Σu_n of positive terms, we can find a function v_n such that $v_n \prec u_n$ and Σv_n is divergent; *i.e.* given any divergent series we can find one more slowly divergent.

Given any convergent series Σu_n of positive terms, we can find v_n so that $v_n \succ u_n$ and Σv_n is convergent; *i.e.* given any convergent series we can find one more slowly convergent.

Given any function $\phi(n)$ tending to infinity, however slowly, we can find a convergent series Σu_n and a divergent series Σv_n such that $v_n/u_n = \phi(n)$.

Given an infinite sequence of series, each converging (diverging) more slowly than its predecessor, we can find a series which converges (diverges) more slowly than any of them.

There is no function $\phi(n)$ such that $u_n\phi(n) \succcurlyeq 1$ is a necessary condition for the divergence of Σu_n, and no function $\phi(n)$ such that $\phi(n) \succ 1$ and $u_n\phi(n) \preccurlyeq 1$ is a necessary condition for the convergence of Σu_n.

If u_n is a steadily decreasing function of n, then $nu_n \prec 1$ is a necessary condition for convergence; but there is no function $\phi(n)$ such that $\phi(n) \succ 1$ and $n\phi(n)\,u_n \prec 1$ is a necessary condition.

If however nu_n decreases steadily, then $n \log n u_n \to 0$ is a necessary condition; and if $n\psi(n) u_n$, where $n\psi(n) \succ 1$ and $\int \frac{dn}{n\psi(n)} \succ 1$, decreases steadily, then

$$\left(n\psi(n) \int \frac{dn}{n\psi(n)}\right) u_n \to 0$$

is a necessary condition.

If Σu_n is divergent, and $U_n = u_1 + u_2 + \ldots + u_n$, then $\Sigma (u_n / U_n)$ is also divergent; and if also $u_n \prec U_n$ then

$$\frac{u_1}{U_1} + \frac{u_2}{U_2} + \ldots + \frac{u_n}{U_n} \sim \log U_n.$$

See Abel, **1**, **2**; Bromwich, **1**, 40; Dini, **1**; Hadamard, **2**; Littlewood, **5**; Pringsheim, **1** (353, 939), **2** (89), **3**, **4**.

For examples of series and integrals which converge or diverge so slowly as not to answer to any of the logarithmic criteria, so that the logarithmic tests are insufficient (§ 4.42), or to which the logarithmic tests are inapplicable, see Borel, **4**, **5**; du Bois-Reymond, **3**, **7**, **8**; Gilbert, **1**; Goursat, **1** (1), 219; Hardy, **3**, (1), (2), (3), (5); Pringsheim, **1** (353), **3** (343), **5**, **6**; Thomae, **2**.

6.23. Asymptotic formulae for finite sums. A closely connected problem is that of the determination of asymptotic formulae for

$$A_n = a_1 + a_2 + \ldots + a_n$$

when the behaviour of a_n for large values of n is known. The principal weapons for dealing with this problem are (i) the theorem of Cauchy and Stolz, that $A_n \sim CB_n$ if Σb_n is a divergent series of positive terms and $a_n \sim Cb_n$, (ii) the 'Euler-Maclaurin sum formula'

$$\sum_1^n f(\nu) = \int^n f(x)\,dx + C + \tfrac{1}{2} f(n) + \frac{B_1}{2!} f'(n) - \frac{B_2}{4!} f'''(n) + \ldots,$$

and in particular (iii) the theorem of Maclaurin and Cauchy that

$$f(1) + f(2) + \ldots + f(n) - \int_1^n f(x)\,dx,$$

where $f(x)$ is a positive decreasing function of x, tends to a limit when $n \to \infty$.

For (i) see Cauchy, **1**, 59; Jensen, 1; Stolz, **1**; and for (iii) Cauchy, **2**; Maclaurin, **1** (1), 289. Proofs of either theorem will be found in any modern text book of analysis or the theory of series; see Bromwich, **1**, 29, 377; Knopp, **1**, 68, 286. For further developments see Bromwich, **2**; Dahlgren, **1**; Hardy, **3** (4), **8**; Nörlund, **1**. The literature of the general Euler-Maclaurin sum formula is too extensive to be summarized here; see Bromwich, **1**, 238, 324; Nörlund, **1**, **2**; Pringsheim, **2**, 102; Seliwanoff, **1**, 929.

Among the most important results which follow from these theorems are

$$1^s + 2^s + \dots + n^s \sim \frac{n^{s+1}}{s+1} \quad (s > -1),$$

$$1^s + 2^s + \dots + n^s - \frac{n^{s+1}}{s+1} \sim \zeta(-s) \quad (-1 < s < 0),$$

and generally

$$\sum_1^n \nu^s - \frac{n^{s+1}}{s+1} - \tfrac{1}{2}n^s - \sum_1 (-1)^{i-1}\binom{s}{2i-1}\frac{B_i}{2i}n^{s-2i+1} \sim \zeta(-s).$$

Here s is positive and not integral, $\zeta(-s)$ is the Zeta-function of Riemann, and the summation with respect to i is continued till we come to a negative power of n. Again

$$1 + \frac{1}{2} + \dots + \frac{1}{n} - \log n \sim A,$$

$$1 + \frac{\alpha . \beta}{1 . \gamma} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 . 2 . \gamma(\gamma+1)} + \dots \text{ to } n \text{ terms},$$

$$\sim \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \frac{n^{\alpha+\beta-\gamma}}{\alpha+\beta-\gamma} \quad (\alpha + \beta > \gamma),$$

or

$$\sim \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \log n \quad (\alpha + \beta = \gamma).$$

In connection with the last result see Bromwich, **4**; in the earlier formula A is Euler's constant.

The most important formula of this kind is

$$\log 1 + \log 2 + \dots + \log n - (n + \tfrac{1}{2})\log n + n \sim \tfrac{1}{2}\log(2\pi),$$

which, in the form

$$n! \sim n^{n+\frac{1}{2}} e^{-n} \sqrt{(2\pi)},$$

constitutes *Stirling's Theorem*. Another formula of the same kind is

$$1^1 2^2 3^3 \dots n^n \sim B n^{\frac{1}{2}n^2 + \frac{1}{2}n + \frac{1}{12}} e^{-\frac{1}{4}n^2},$$

where B is a constant defined by the equation

$$\log B = \tfrac{1}{12}\log 2\pi + \tfrac{1}{12}\gamma + \frac{1}{2\pi^2}\sum_1^\infty \frac{\log \nu}{\nu^2} = \frac{\gamma + \ln 2\pi}{12} - \frac{\zeta'(2)}{2\pi^2}$$

The literature of Stirling's Theorem is also very extensive; see Bromwich, **1**, 461; Brunel, **1**; Nielsen, **1**, 92; Whittaker and Watson, **1**, 251, 276. As regards the constant B see Barnes, **1**; Glaisher, **1**, **2**; Kinkelin, **1**.

6.24. A proof of Stirling's Theorem. Stirling's Theorem, as stated in § 6.23, may be proved in an almost elementary manner*; but

* For such a proof see, *e.g.*, Cesàro, **1**, 221, 395; Jolliffe, **1**. The principal difficulty of an elementary proof is naturally the determination of the constant $\sqrt{(2\pi)}$.

it will be more instructive here to give a proof depending on the representation of $\Gamma(n+1)$ as a definite integral. The method employed, the principle of which may be traced back to Laplace*, is that which, when extended to the field of the complex variable, is known as the 'Methode der Sattelpunkte' or 'method of steepest descents'†.

We suppose n positive and large, but not necessarily integral. In the integral

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n \, dx$$

the maximum of the integrand occurs for $x = n$. We therefore write

(6.241)

$$\Gamma(n+1) = J = \int_0^\infty = \int_0^{(1-\eta)n} + \int_{(1-\eta)n}^{(1+\eta)n} + \int_{(1+\eta)n}^{2n} + \int_{2n}^\infty = J_1 + J_2 + J_3 + J_4,$$

say, where $0 < \eta < 1$.

In J_1 and J_3 we write $x = n(1-y)$ and $x = n(1+y)$ respectively. Observing that the functions $e^y(1-y)$ and $e^{-y}(1+y)$ each decrease steadily as y increases from 0 to 1, we obtain

(6.242)

$$J_1 = n^{n+1} e^{-n} \int_\eta^1 e^{ny} (1-y)^n \, dy < n^{n+1} e^{-n} \{e^\eta (1-\eta)\}^n = n^{n+1} e^{-n} E_1^n,$$

(6.243)

$$J_3 = n^{n+1} e^{-n} \int_\eta^1 e^{-ny} (1+y)^n \, dy < n^{n+1} e^{-n} \{e^{-\eta} (1+\eta)\}^n = n^{n+1} e^{-n} E_3^n,$$

say; here E_1 and E_3 are less than 1. And if we apply the same transformation to J_4 as to J_3, and observe that $e^{-y}(1+y)$ also decreases from $y = 1$ onwards, we find

(6.244)

$$J_4 = n^{n+1} e^{-n} \int_1^\infty e^{-ny} (1+y)^n dy < n^{n+1} e^{-n} \left(\frac{2}{e}\right)^{n-1} \int_1^\infty e^{-y} (1+y) \, dy$$

$$= \frac{3}{2} e n^{n+1} e^{-n} \left(\frac{2}{e}\right)^n.$$

From (6.242), (6.243), and (6.244) it follows that

(6.245) $$\lim_{n \to \infty} n^{-n-\frac{1}{2}} e^n (J_1 + J_3 + J_4) = 0.$$

In J_2 we write again $x = n(1+y)$. We have then

$$n \log x - x = n \log n - n - ny + n \log(1+y) = n \log n - n - \frac{ny^2}{2(1+\theta y)^2},$$

where $-1 < \theta < 1$. Hence J_2 lies between

$$n^{n+1} e^{-n} \int_{-\eta}^{\eta} e^{\left\{-\frac{ny^2}{2(1-\eta)^2}\right\}} dy = (1-\eta) n^{n+\frac{1}{2}} e^{-n} \sqrt{2} \int_{-\zeta}^{\zeta} e^{-w^2} dw,$$

* Laplace, 1, 88. † Watson, 1, 235.

where
$$\zeta = \frac{\eta}{1-\eta}\sqrt{\left(\frac{n}{2}\right)},$$
and the corresponding expression in which $1-\eta$ is replaced by $1+\eta$. The limit of the integral when n, and therefore ζ, tends to infinity is $\surd\pi$. Hence

(6.246)
$$\varliminf_{n\to\infty} n^{-n-\frac{1}{2}}e^{n}J_2 \geqq (1-\eta)\surd(2\pi), \quad \varlimsup_{n\to\infty} n^{-n-\frac{1}{2}}e^{n}J_2 \leqq (1+\eta)\surd(2\pi).$$

From (6.245) and (6.246) it follows that
$$(1-\eta)\surd(2\pi) \leqq \varliminf n^{-n-\frac{1}{2}}e^{n}J \leqq \varlimsup n^{-n-\frac{1}{2}}e^{n}J \leqq (1+\eta)\surd(2\pi).$$
But η is arbitrary, and J is independent of η. Hence

(6.247)
$$\lim n^{-n-\frac{1}{2}}e^{n}J = \surd(2\pi),$$
which is Stirling's Theorem.

As a corollary we note that

(6.248)
$$\frac{\Gamma(n+a)}{\Gamma(n+b)} \sim \frac{(n+a)^{n+a-\frac{1}{2}}}{(n+b)^{n+b-\frac{1}{2}}}e^{b-a} \sim n^{a-b}.$$

6.25. A general result for L-functions. The results of Section 5 enable us to obtain a general formula for A_n whenever a_n is an L-function of n and Σa_n is divergent.

Theorem 33. *If a_n is the value for $x=n$ of an L-function $a(x)$, and Σa_n is divergent, then*

(6.251)
$$A_n \sim a(n)$$
if $a(x) \succ e^{\Delta x}$,

(6.252)
$$A_n \sim \int_1^n a(x)\,dx$$
if $a(x) \prec e^{\delta x}$, and

(6.253)
$$A_n \sim \frac{a}{1-e^{-a}}\int_1^n a(x)\,dx$$
if $a(x) = e^{ax}b(x)$, where $e^{-\delta x} \prec b(x) \prec e^{\delta x}$.

Suppose first that $a(x) \succ e^{\Delta x}$, so that $a' \succ a$. Then, if we suppose, as we may do without loss of generality, that $a(x)$ increases from $x=1$, we have
$$A_{n-1} = \sum_1^{n-1} a(\nu) < \int_1^n a(x)\,dx \sim \frac{\{a(n)\}^2}{a'(n)} \prec a(n),$$
by Theorem 25. Hence $A_n \sim a(n)$.

Next, suppose $a(x) \prec e^{\delta x}$, so that $a' \prec a$. Then
$$a_\nu - c_\nu = a(\nu) - \int_{\nu-1}^{\nu} a(x)\,dx = \int_{\nu-1}^{\nu}\{a(\nu)-a(x)\}\,dx = \int_{\nu-1}^{\nu}(\nu-x)a'(r)\,dx,$$

where $\nu - 1 < r < \nu$. But $a'(r) \sim a'(\nu)$, by Theorem 31; and so

$$a_\nu - c_\nu \preccurlyeq a'(\nu) \prec a(\nu) = a_\nu.$$

It follows that $a_\nu \sim c_\nu$ and $A_n \sim C_n$, which is (6.252).

Finally suppose $a(x) = e^{\alpha x} b(x)$. Then

$$\int_{\nu-1}^{\nu} \{a(\nu) - a(x)\}\, dx = b(\nu) \int_{\nu-1}^{\nu} (e^{\alpha\nu} - e^{\alpha x})\, dx + \int_{\nu-1}^{\nu} e^{\alpha x} \{b(\nu) - b(x)\}\, dx$$

$$= \beta_\nu + \gamma_\nu,$$

say. Here $b(\nu) - b(x) = (\nu - x)\, b'(r) \preccurlyeq b'(\nu)$, and so

$$\gamma_\nu \preccurlyeq e^{\alpha\nu} b'(\nu) \prec e^{\alpha\nu} b(\nu) = a_\nu,$$

while

$$\beta_\nu = \left(1 - \frac{1 - e^{-\alpha}}{\alpha}\right) a(\nu).$$

Hence

$$a_\nu - c_\nu \sim 1 - \frac{1 - e^{-\alpha}}{\alpha} a_\nu, \quad a_\nu \sim \frac{\alpha}{1 - e^{-\alpha}} c_\nu;$$

and (6.253) follows.

It is also possible, by using Theorem 26, to obtain formulae for A_n when $a_n = f_n e^{i\phi_n}$, where f and ϕ are L-functions subject to certain limitations; but the results are more complicated and less general. It is easy to see that comprehensive results are not to be expected here. The series $\Sigma\, e^{ain^2}$, for example, behaves in a very intricate manner, depending on the arithmetic nature of the number a*. But, if the increase of f_n and ϕ_n is sufficiently slow, A_n will behave like the integral $\int^n f(u)\, e^{i\phi(u)}\, du$, and the series $\Sigma\, a_n$ will be convergent if $f \prec \phi'$.

6.26. Formulae involving prime numbers and arithmetical functions. It is known that, if $\pi(n)$ is the number of prime numbers not exceeding n, and p_n is the nth prime, so that $\pi(n)$ and p_n are inverse functions, then

(6.261) $$\pi(n) \sim \frac{n}{\log n}, \quad p_n \sim n \log n.$$

More precisely

(6.262) $$\pi(n) = \int_2^n \frac{dt}{\log t} + O\left(ne^{-A\sqrt{(ln\, lln)}}\right) = \operatorname{Li} n + O\left(ne^{-A\sqrt{(ln\, lln)}}\right)\dagger,$$

where $A > 0$. If the hypothesis of Riemann concerning the zeros of the Zeta-function $\zeta(s)$ is true, the error term may be replaced by $O(n^{\frac{1}{2}+\delta})$ and indeed by $O(\sqrt{n}\, ln)$. On the other hand the order of the error is certainly not less than

$$O\left(\frac{\sqrt{n}\, l_3 n}{ln}\right)\ddagger.$$

* Hardy and Littlewood, **3** (2). For a discussion of the series $\Sigma n^{-b} e^{Ain^a}$, where $0 < a < 1$, see Hardy, **8**.

† The classical formula has an error term $O\left\{ne^{-A\sqrt{(ln)}}\right\}$. For the more precise result stated here see Landau, **5**, **6**; Littlewood, **7**.

‡ Littlewood, **6**; Hardy and Littlewood, **4**.

It is easily proved by partial integration that

(6.263) $$\int_2^n \frac{dt}{\log t} = \frac{n}{ln} + \frac{n}{(ln)^2} + \frac{2!n}{(ln)^3} + \dots + \frac{(k-1)!n}{(ln)^k} + O\left\{\frac{n}{(ln)^{k+1}}\right\}$$

for every value of k; while $e^{-A\sqrt{(ln\, lln)}}$ tends to zero more rapidly than any power of ln. Hence the right hand side of (6.263) is a genuine approximation to $\pi(n)$ for any value of k.

The order of magnitude of a sum of the form

$$\sum_{p<n} f(p)$$

may, with certain reservations, be found by replacing the nth prime by $n \log n$. Thus

$$\sum_{p<x} \frac{1}{p} \sim l_2 x, \quad \sum_{p<x} \frac{lp}{p} \sim lx, \quad \sum_{p<x} lp \sim x,$$

while $\Sigma \frac{1}{plp}$ is convergent. For a comprehensive account of the theory see Landau, 1.

We quote some additional examples of asymptotic formulae for arithmetical functions. We write $\pi_\nu(x)$ for the number of numbers, less than x, composed of just ν factors (repeated or not); $Q(x)$ for the number of numbers with no repeated factor; $R(x)$ for the number of numbers of the form $2^{a_2}3^{a_3}\dots p^{a_p}$, where $a_2 \geqq a_3 \geqq \dots$; $p(n)$ for the number of partitions of n; and $p_r(n)$ for the number of partitions of n into perfect rth powers. Then

$$\pi_\nu(x) \sim \frac{1}{(\nu-1)!}\frac{x(llx)^{\nu-1}}{lx} *, \quad Q(x) \sim \frac{6x}{\pi^2} \dagger,$$

$$\log R(x) \sim \frac{2\pi}{\sqrt{3}}\sqrt{\left(\frac{lx}{llx}\right)} \ddagger, \quad p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{(\frac{2}{3}n)}} \S,$$

$$p_r(n) \sim (2\pi)^{-\frac{1}{2}(r+1)} \sqrt{\left(\frac{r}{r+1}\right)}\, kn^{\frac{1}{r+1}-\frac{3}{2}} e^{\left\{(r+1)\, kn^{\frac{1}{r+1}}\right\}},$$

where $$k = \left\{\frac{1}{r}\Gamma\left(1+\frac{1}{r}\right)\zeta\left(1+\frac{1}{r}\right)\right\}^{\frac{r}{r+1}}$$

and $\zeta(s)$ is Riemann's Zeta-function.

6.31. Power-series. The theory of integral functions. The radius of convergence R of a power-series

(6.311) $$f(x) = \Sigma a_n x^n$$

is given || by

$$\frac{1}{R} = \overline{\lim_{n\to\infty}}\, |a_n|^{\frac{1}{n}}.$$

The series is convergent for all values of x if $\sqrt[n]{|a_n|} \to 0$, *i.e.* if $|a_n| \prec e^{-\Delta n}$. In this case $f(x)$ is called an *integral function*.

* Landau, 1, 208, 211.
† Landau, 1, 582.
‡ Hardy and Ramanujan, 1.
§ Hardy and Ramanujan, 2.
|| See, *e.g.*, Goursat, 1 (1), 443.

The three most important characteristics of an integral function are measured by

(i) $\alpha_n = | a_n |$, the modulus of the nth coefficient;

(ii) $M(r)$, the maximum of $|f(x)|$ on the circle $|x| = r$;

(iii) $\gamma_n = | c_n |$, the modulus of the nth zero, in order of absolute magnitude.

It is known that $M(r)$ is a steadily increasing function of r, and that $M(r) \succ r^{\Delta}$, except in the trivial case (which we ignore) in which $f(x)$ is a polynomial*. A function for which $M(r) \prec e^{r^{\Delta}}$ is called a function *of finite order*, and we shall consider such functions only.

The principal problem of the theory is to determine the relations between the rates of increase of $1/\alpha_n$, $M(r)$, and γ_n. Those which hold between the first two functions are the simplest, and we shall confine our attention to them. The theory of γ_n is complicated by the 'Picard case of exception', arising from functions which, like e^x, have no zeros, or whose zeros are scattered abnormally over the plane. The increases of the three functions may be measured by 'indices' defined as follows†.

The μ-index μ of $f(x)$ is the greatest number ξ such that

(6.312) $$\sqrt[n]{\alpha_n} < n^{-\xi+\epsilon}$$

for every positive ϵ and all sufficiently large values of n. It is plain that $\mu \geqq 0$, since $\alpha_n \rightarrow 0$. It may happen that (6.312) is true for all values of μ; in this case we say that the μ-index is infinite. The ν-index ν is the least number ξ such that

(6.313) $$M(r) < e^{r^{\xi+\epsilon}}$$

for every positive ϵ and all sufficiently large values of r. The ρ-index ρ is the least number ξ such that

$$\Sigma \frac{1}{\gamma_n^{\xi+\epsilon}}$$

is convergent for every positive ϵ. In particular these conditions are satisfied if

$$n^{-\mu-\delta} \prec \sqrt[n]{\alpha_n} \prec n^{-\mu+\delta},\ e^{r^{\nu-\delta}} \prec M(r) \prec e^{r^{\nu+\delta}},\ n^{\frac{1}{\rho}-\delta} \prec \gamma_n \prec n^{\frac{1}{\rho}+\delta},$$

or if

$$l\left(\frac{1}{\alpha_n}\right) \sim \mu n\, ln,\ l_2 M(r) \sim \nu\, lr,\ l\gamma_n \sim \frac{ln}{\rho}.$$

* For the second proposition see Goursat, **1** (2), 92. It is curiously difficult to give a reference to a direct and explicit proof of the first. It is included implicitly in one of the classical proofs of the fundamental theorem of algebra (see, *e.g.*, Hardy, **1**, 433) and in the familiar theorem that a potential function cannot have a maximum at a point of regularity.

† Vivanti, **1**, 228.

The fundamental theorems of the subject are (i) that $\nu = 1/\mu$, it being understood that this means $\nu = 0$ when the μ-index is infinite, and (ii) that, with certain reservations, $\rho = \nu$.

6.32. Proof* that $\mu = 1/\nu$. (i) We suppose $\mu > 0$, and we prove first that $\nu \geqslant 1/\mu$. We denote by $\mathbf{M}(r)$ the maximum of $a_n r^n$ for $n = 0, 1, 2, \ldots$. Then† $M(r) > \mathbf{M}(r)$ for all values of r. It follows from the definition of μ that

(6.321) $$\sqrt[n]{a_n} > n^{-\mu-\epsilon}$$

for every positive ϵ and an indefinitely increasing sequence (n_j) of values of n. If n has one of these values,

(6.322) $$a_n r^n > (rn^{-\mu-\epsilon})^n.$$

The right hand side, considered as a function of a continuous variable n, attains a maximum

$$e^{\left(\frac{\mu+\epsilon}{e} r^{\mathbf{m}}\right)},$$

where $\mathbf{m} = 1/(\mu + \epsilon)$, for

(6.323) $$n = r^{\mathbf{m}}/e.$$

If r has such a value that (6.323) is one of the integers n_j, then

$$M(r) > \mathbf{M}(r) > e^{\left(\frac{\mu+\epsilon}{e} r^{\mathbf{m}}\right)}.$$

This is true for a sequence of values of r surpassing all limit, and $\mathbf{m}$ is any number less than $1/\mu$. It follows that $\nu \geqq 1/\mu$.

(ii) To obtain an upper bound for $M(r)$, we observe that

$$r\sqrt[n]{a_n} < rn^{-\mu+\epsilon} < \tfrac{1}{2}$$

if

(6.324) $$n \geqq n_r = (2r)^{\mathbf{m}'},$$

where $\mathbf{m}' = 1/(\mu - \epsilon)$, and r is large enough. Thus

(6.325) $$M(r) \leqq \sum_0^{n_r-1} a_n r^n + \sum_{n_r}^{\infty} a_n r^n < n_r \mathbf{M}(r) + \sum_0^{\infty} 2^{-n}$$
$$= n_r \mathbf{M}(r) + 2 < 2n_r \mathbf{M}(r),$$

if r is large enough. But

(6.326) $$\mathbf{M}(r) = \operatorname{Max} a_n r^n \leqq \operatorname{Max} (rn^{-\mu+\epsilon})^n = e^{\left(\frac{\mu-\epsilon}{e} r^{\mathbf{m}'}\right)}.$$

From (6.324), (6.325), and (6.326) it follows that

$$M(r) < 2\,(2r)^{\mathbf{m}'} e^{\left(\frac{\mu-\epsilon}{e} r^{\mathbf{m}'}\right)}$$

for all sufficiently large values of r. Here $\mathbf{m}'$ is any number greater than $1/\mu$. Hence $\nu \leqq 1/\mu$, and so $\nu = 1/\mu$.

* The proof is modelled on that given by Lindelöf, **3**. † Goursat, **1** (2), 92.

The fundamental idea of the proof which precedes is that *the increase of $f(x)$ is measured*, with sufficient accuracy for the determination of the indices, *by that of its greatest term*. In the exponential series, for example, the greatest term is that for which $n=[x]$, and the increase of this term is $e^x/\sqrt{x}$.

We have assumed μ positive and finite. A slight variation of the argument shows (*a*) that $\nu=0$ when μ is infinite, and (*b*) that $f(x)$ is not of finite order when $\mu=0$.

6.33. Special results. If we make more drastic assumptions about the coefficients a_n, we can naturally obtain more precise results about $f(x)$. Thus if

$$\{n\,(ln)^{-b_1}\ldots(l_kn)^{-b_k+\delta}\}^{-1/\nu}\prec\sqrt[n]{a_n}\prec\{n\,(ln)^{-b_1}\ldots(l_kn)^{-b_k-\delta}\}^{-1/\nu},$$

then $$e\,\{r^\nu\,(lr)^{b_1}\ldots(l_kr)^{b_k-\delta}\}\prec M(r)\prec e\,\{r^\nu\,(lr)^{b_1}\ldots(l_kr)^{b_k+\delta}\},$$

and conversely. If

$$\sqrt[n]{a_n}=n^{-1/\nu}\,\lambda(n),$$

where $$e^{-\delta\sqrt{ln}}\prec\lambda(n)\prec e^{\delta\sqrt{ln}},$$

then $$\log f(x)\sim\frac{1}{\nu e}\,\{x\lambda(x^\nu)\}^\nu.$$

As examples of still more accurate and special results we may quote the following:

$$\Sigma\,\frac{x^n}{n^{an}}\sim\sqrt{\left(\frac{2\pi}{ea}\right)}\,x^{1/2a}\,e^{(a/e)\,x^{1/a}},$$

$$\Sigma\,\frac{x^n}{(n\,!)^a}\sim\frac{1}{\sqrt{a}}\,(2\pi)^{(1-a)/2}\,x^{(1-a)/2a}\,e^{ax^{1/a}},\quad\Sigma\,\frac{x^n}{\Gamma(an+1)}\sim\frac{1}{a}\,e^{x^{1/a}},$$

$$\Sigma\,e^{-n^p}\,x^n\sim\sqrt{\left\{\frac{2\pi}{p\,(p-1)}\right\}}\left(\frac{\log x}{p}\right)^{\frac{2-p}{2p-2}}e^{(p-1)\left(\frac{\log x}{p}\right)^{p/(p-1)}},$$

where $a>0$ and in the last formula $1<p<2$, and $x\to\infty$ by positive values. These results may of course be used to give an upper limit for the modulus of the particular function considered when x is not necessarily real, and so for $M(r)$.

General accounts of the theory of integral functions are given by Borel, **2**; Vivanti, **1**; Bieberbach, **1**; Valiron, **1**. The second edition of the first work contains a very valuable note by Valiron on the latest developments of the theory, and the second work a very complete bibliography up to 1906. Particularly important memoirs (beyond those on which Borel's account of the theory is based) are those of Boutroux, **1**; Lindelöf, **2**; Pringsheim, **7**; Valiron, **2**, **3**; and Wiman, **1**, **2**, **3**. For more precise and special developments, such as those quoted at the beginning of this section, see in particular Le Roy, **1**; Lindelöf, **3**; Littlewood, **1**, **2**, **3**, **4**; and Mellin, **1**. For the theory of integral functions of infinite order, see Blumenthal, **1**.

6.34. Irregularly increasing functions defined by power series. Power series with gaps. The theory of integral functions suggests a method of much interest for the construction of irregularly increasing functions.

Suppose that $\phi(x) = \Sigma a_n x^n$ is an integral function with positive and decreasing coefficients, and that, for a given x, $\varpi(x) = a_\nu x^\nu$ is the greatest term of the series. In general one term will be the greatest, but for particular values of x, say $\xi_1, \xi_2, \ldots$, two consecutive terms will be equal*.

As x increases, the index ν of $\varpi(x)$ increases, and tends to infinity with n: it thus defines a function $\nu(x)$ such that

$$\nu(x) = i \quad (\xi_i < x < \xi_{i+1}).$$

At the point of discontinuity ξ_i, where $\nu(x)$ jumps from $i-1$ to i, we may assign to it the value i. When ν is thus defined for all values of x, $\varpi(x)$ defines a function of x which tends continuously and steadily to infinity with x; and it may be expected that the increase of ϖ will give a fair approximation to that of ϕ.

Now let
$$f(x) = \Sigma a_{\chi(n)} x^{\chi(n)},$$
where $\chi(n) \succ n$; and let $p(x)$ be the function related to f as $\varpi(x)$ is to ϕ. The laws of increase of $\varpi(x)$ and of $p(x)$ may be expected to be very much the same, for $p(x)$ is defined by a selection from *some* of the terms from *all* of which $\varpi(x)$ was selected. The increase of $f(x)$ clearly cannot be greater, and may be expected to be less, than that of $\phi(x)$; but it cannot be less than that of $p(x)$. Hence we may expect relations of the type

$$p \asymp \varpi \prec f \prec \phi.$$

The more rapidly we suppose $\chi(n)$ to increase, the lower in the gap between ϖ and ϕ will f sink, and, if we suppose χ to increase with sufficient rapidity, we may expect to find that $\varpi \asymp f$, so that the increase of f is completely dominated by that of one variable term. We shall then have

$$f(x) \asymp a_{N(x)} x^{N(x)},$$

where $N(x)$ is a function of x which assumes successively each of a series of integral values N_i, so that

$$N(x) = N_i, \quad (x_i \leqq x < x_{i+1}).$$

But, as x increases from x_i to x_{i+1}, the order of $a_{N_i} x^{N_i}$, considered as a function of x, may vary considerably, since N_i, though depending on the interval (x_i, x_{i+1}), does not depend on the particular position of x in that interval. We are thus likely to be led to functions whose increase is irregular in the sense explained in § 4.41.

Suppose, for example, that $a_n = n^{-n}$, so that (§ 6.33)

$$\phi(x) = \Sigma \left(\frac{x}{n}\right)^n \sim \sqrt{\left(\frac{2\pi x}{e}\right)} e^{x/e}.$$

Here
$$\xi_i = i\left(1 + \frac{1}{i}\right)^{i+1} \sim ei,$$
and it is easily shown that $\varpi(x) \asymp e^{x/e}$.

Now let $\chi(n) = 2^n$, so that

$$f(x) = \Sigma \frac{x^{2^n}}{2^{n2^n}} = \Sigma v_n,$$

* We ignore the possibility of more than two terms being equal.

say. Then $v_{i-1}=v_i$ if $x=2^{i+1}$, so that $x_i=2^{i+1}$ and $N_i=2^i$ for

$$2^{i+1} \leqq x < 2^{i+2}.$$

For this range of values of x, v_i is the greatest term; when $x=2^{i+2}$, $v_i=v_{i+1}$. Further, it is not difficult to show that $f(x) \asymp p(x)=v_i$, the behaviour of $f(x)$ being dominated by that of its greatest term*. If we put $x=2^{i+1+\theta}$, where $0<\theta<1$, we find

$$f(x) \asymp v_i = 2^{(1+\theta)\,2^i} = 2^{ax},$$

where $a=(1+\theta)\,2^{-1-\theta}$. This is a maximum when $1+\theta=1/(\log 2)$, when it is equal to $1/(e\log 2)=\cdot 53\ldots$. Hence the increase of $f(x)$ oscillates (roughly) between those of $2^{\cdot 53\ldots x}$ and $2^{\frac{1}{2}x}$.

Another example of an irregularly increasing function defined in a similar manner is

$$f(x)=\Sigma \frac{x^{n^3}}{(n^3)!},$$

the increase of which oscillates between the increases of $e^x/\sqrt{x}$ and

$$x^{-\frac{1}{2}} e^{x-\frac{3}{8}x^{1/3}} \dagger.$$

These examples are of course typical of a large class of functions.

6.35. Power-series with a finite radius of convergence. When the radius of convergence of the power-series (6.311) is finite, it may be supposed, without loss of generality, to be 1. The necessary and sufficient condition for this is that $\overline{\lim} \sqrt[n]{|a_n|}=1$; this is true in particular if a_n is positive and $e^{-\delta n} \prec a_n \prec e^{\delta n}$.

Suppose in particular that a_n is positive and that Σa_n is divergent, so that $f(x) \rightarrow \infty$ when $x \rightarrow 1$ ‡. Then a large number of important theorems have been proved which embody relations between (*a*) the increase of $A_n=a_0+a_1+\ldots+a_n$ as $n \rightarrow \infty$ and (*b*) the increase of $f(x)$ as $x \rightarrow 1$.

The most fundamental theorem is

Theorem 34. *If a_n and b_n are positive, and $A_n \sim B_n$, then*

(6.351) $$f(x)=\Sigma a_n x^n \sim g(x)=\Sigma b_n x^n.$$

In particular this is so if $a_n \sim b_n$§.

* We may say roughly that *in general* $f \sim p$, that is to say, $f/p \rightarrow 1$ as $x \rightarrow \infty$ through any sequence of values not falling inside any of certain intervals, as small as we please, surrounding the values ξ_i. At a point ξ_i, f/p is nearly equal to 2.

† Hardy, **3** (3).

‡ Bromwich, **1**, 130

§ Bromwich, **1**, 132. The theorem is due to Cesàro, **2**.

We have

$$F(x)=\frac{f(x)}{1-x}=\Sigma A_n x^n, \quad G(x)=\frac{g(x)}{1-x}=\Sigma B_n x^n,$$

and it is enough to prove that $F(x)\sim G(x)$.

Given any positive ϵ, we have $B_n(1-\epsilon)<A_n<B_n(1+\epsilon)$ for $n\geqq N(\epsilon)$, say; and

$$F(x)=\sum_0^{N-1}A_n x^n+\sum_N^\infty A_n x^n=F_N(x)+\sum_N^\infty A_n x^n$$

lies between

$$F_N(x)+(1-\epsilon)\sum_N^\infty B_n x^n, \quad F_N(x)+(1+\epsilon)\sum_N^\infty B_n x^n;$$

and therefore between

$$-B_N+(1-\epsilon)\,G(x), \quad A_N+(1+\epsilon)\,G(x).$$

Hence

$$1-\epsilon\leqq\varliminf_{x\to 1}\frac{F(x)}{G(x)}\leqq\varlimsup_{x\to 1}\frac{F(x)}{G(x)}\leqq 1+\epsilon$$

for every positive ϵ, which proves the theorem.

We have, for example,

$$\frac{\Gamma(1-p)}{(1-x)^{1-p}}=\sum_{n=0}^\infty\frac{\Gamma(n+1-p)}{\Gamma(n+1)}x^n=\Sigma b_n x^n,$$

say, and $b_n\sim n^{-p}=a_n$, by (6.248)*. Hence

$$\Sigma\frac{x^n}{n^p}\sim\frac{\Gamma(1-p)}{(1-x)^{1-p}} \qquad (p<1).$$

Similarly

$$F(\alpha,\beta,\gamma,x)\sim\frac{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}\frac{1}{(1-x)^{\alpha+\beta-\gamma}} \quad (\alpha+\beta>\gamma),$$

$$F(\alpha,\beta,\alpha+\beta,x)\sim\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\,l\left(\frac{1}{1-x}\right).$$

Of further results the following is typical: if

$$a_n\sim n^{-p}\{ln\ldots l_{m-1}n\,(l_m n)^q\ldots(l_{m+k}n)^{q_k}\}^{-1},$$

then

$$f(x)\sim\frac{\Gamma(1-p)}{(1-x)^{1-p}}\left\{l\,\frac{1}{1-x}\ldots l_{m-1}\frac{1}{1-x}\left(l_m\frac{1}{1-x}\right)^q\ldots\left(l_{m+k}\frac{1}{1-x}\right)^{q_k}\right\}^{-1}$$

if $p<1$, $q\neq 1$: but

$$f(x)\sim\frac{1}{1-q}\left(l_m\frac{1}{1-x}\right)^{1-q}\left\{\left(l_{m+1}\frac{1}{1-x}\right)^q\ldots\left(l_{m+k}\frac{1}{1-x}\right)^{q_k}\right\}^{-1}$$

if $p=1$, $q<1$. Thus

$$\Sigma\frac{x^n}{n^p\,(lx)^q}\sim\frac{\Gamma(1-p)}{(1-x)^{1-p}}\left(l\,\frac{1}{1-x}\right)^{-q} \quad (p<1).$$

* Appell, 1.

As specimens of further results of this character we may quote

$$x+x^4+x^9+\dots \sim \tfrac{1}{2}\sqrt{\left(\frac{\pi}{1-x}\right)},$$

$$x+x^a+x^{a^2}+\dots \sim \frac{1}{la}\, l\left(\frac{1}{1-x}\right) \qquad (a>1),$$

$$\Sigma a^n x^{n^2} \sim e\left\{\tfrac{1}{4}\frac{(la)^2}{l\,(1/x)}\right\} \qquad (a>1),$$

$$\Sigma\, e^{n/ln}\, x^n = e_2\{u/(1-x)\} \qquad (u\sim 1).$$

Many similar results have been established about series other than power series: thus

$$\Sigma\, \frac{x^n}{n\,(1+x^n)} \sim \tfrac{1}{2}l\left(\frac{1}{1-x}\right),$$

$$\Sigma\, \frac{x^n}{1-x^n} \sim \frac{1}{1-x}\, l\left(\frac{1}{1-x}\right).$$

As an example of a more precise result we may quote the formula

$$\Sigma\, \frac{x^n}{1+x^{2n}} = \tfrac{1}{4}\left\{\frac{\pi}{l\,(1/x)} - 1\right\} + O\,\{(1-x)^{\Delta}\}.$$

For accounts of these results, and extensions in various directions, see Barnes, **2**; Borel, **4**; Bromwich, **2**; Hardy, **12**; Knopp, **2**, **3**, **4**; Landau, **4**; Lasker, **1**; Le Roy, **1**; Pringsheim, **8**.

6.41. The increase of real solutions of algebraic differential equations. Suppose that the differential equation

$$(6.411) \qquad f(x, y, y') \equiv \Sigma A x^m y^n y'^p = 0$$

possesses a solution $y = y(x)$ which is real and continuous for $x > x_0$. The problem is to specify as completely as possible the various ways in which y may behave as $x \to \infty$.

This problem was first attacked by Borel (**7**), who proved that the equation cannot have a solution y such that

$$y > e^{e^x} = e_2(x)$$

for values of x surpassing all limit. Borel also stated the corresponding theorem for equations of the second order, viz. that no continuous solution can exceed $e_3(x)$ for values of x surpassing all limit; but his proof is incomplete, and no rigorous proof has yet been found, though there can be little doubt of the truth either of this or the corresponding general theorem for equations of any order.

Later Lindelöf (**1**) returned to the questions raised by Borel, and proved a much more precise result, viz.: *if the equation* (6.411) *is of degree m in x, then*

$$y < e^{\Delta x^{m+1}}$$

for some Δ *and for* $x > x_0$. Further, he proved that either $|y| < e^{x^\delta}$ for

$x > x_0$, or $e^{x^{\rho-\delta}} < |y| < e^{x^{\rho+\delta}}$ for a positive ρ and for $x > x_0$. The solutions of the first class may oscillate, but those of the second are ultimately monotonic, together with all their derivatives.

It is possible to prove a good deal more than this about the equation (6.411)*. Here however we consider only the special equation

(6.412) $$y' = P(x, y)/Q(x, y),$$

where P and Q are polynomials. We prove first that y' is ultimately of constant sign, so that *every solution is ultimately monotonic.*

Suppose the contrary. Then the curves $y = y(x)$, $P = 0$ intersect at points corresponding to an infinity of values of x surpassing all limit. But $P = 0$ consists of a finite number of branches, and so $y = y(x)$ must intersect at least one of these infinitely often.

Now the branches of $P = 0$, which extend to infinity in the direction of the axis of x, consist of (i) a finite number of straight lines $y = c_s$, and (ii) a finite number of branches $y = Y_t(s)$ along which y ultimately increases or decreases. And, in the first place, $y = y(x)$ cannot cut $y = Y_t(x)$ infinitely often. For suppose, for example, that Y_t is ultimately increasing, and that R and S are two successive points of intersection†. Then $y = y(x)$ crosses $y = Y_t(x)$ at R and S, and in each case from above to below, and this is plainly impossible.

We have next to consider the possible intersections of $y = y(x)$ with the straight lines (i), and we may suppose x so large that all intersections with branches (ii) have already been exhausted, so that y' can vanish only at the intersections we are considering. Then y cannot have a maximum or minimum; for at such a point y' would change sign, while P would not, since the line (i) through the point would be the tangent to the point. Hence $y = y(x)$ crosses the tangent and, having crossed it, it cannot return to it without passing through a maximum or minimum. It follows that there is at most a finite number of the intersections in question. Thus y is ultimately monotonic.

6.42. We can go further and prove the following lemma.

Lemma. *Any rational function*

$$H(x, y) = K(x, y)/L(x, y)$$

is ultimately monotonic along the curve $y = y(x)$, *unless* $L = 0$ *is a solution of the equation* (6.411).

We have

$$\frac{dH}{dx} = \frac{\partial H}{\partial x} + \frac{P}{Q}\frac{\partial H}{\partial y} = \frac{U}{W},$$

* Hardy, **10**. See also Boutroux, **1**, 217.

† There must be *successive* points, for all intersections are isolated: see Hardy, **10**.

where U and W are polynomials, and d/dx implies differentiation along the curve $y=y(x)$. If dH/dx is not ultimately of constant sign on the curve, it must vanish or become infinite infinitely often on it. In the first case the curve must have an infinity of intersections with at least one of the finite number of branches of $U=0$. This branch C may, for sufficiently large values of x, be represented in the form

$$(6.421) \qquad y = A_0 x^{a_0} + A_1 x^{a_1} + \ldots,$$

a convergent series of (not generally integral) descending powers of x; and, if $\delta/\delta x$ refers to differentiation along C, then

$$(6.422) \qquad y_1 = \frac{\delta y}{\delta x} = A_0 a_0 x^{a_0 - 1} + A_1 a_1 x^{a_1 - 1} + \ldots.$$

Again, along C, $R(x, y)$ is an algebraic function of x, which may, for sufficiently large values of x, be expressed in the form

$$(6.423) \qquad R = B_0 x^{\beta_0} + B_1 x^{\beta_1} + \ldots,$$

another series of descending powers; and, unless the series (6.422), (6.423) are identical, we shall have $y_1 > R$ or $y_1 < R$ at all points of C from some definite point onwards. From this it follows that, at the points of intersection, C always crosses $y=y(x)$ from one and the same side to the other and the same side, which is plainly impossible.

On the other hand, if the series (6.422) and (6.423) are identical, we have $y_1 = R$, and $U=0$ is a solution of (6.411). In other words, H is constant along $y=y(x)$.

There remains only the possibility that

$$\frac{dH}{dx} = \left(L\frac{dK}{dx} - K\frac{dL}{dx}\right)\Big/ L^2$$

should become infinite infinitely often, as we describe $y=y(x)$. This cannot be true owing to K or L or

$$\frac{dK}{dx} = \frac{\partial K}{\partial x} + \frac{\partial K}{\partial y}\frac{dy}{dx}$$

or dL/dx becoming infinite, and so can only occur if L vanishes infinitely often. But then we can show as above that $L=0$ is a solution of the equation (6.411). Thus the proof of the lemma is completed. As a corollary we see that *any rational function $H(x, y, y')$ is ultimately monotonic, unless its denominator vanishes identically in virtue of* (6.411).

6.43. We can now obtain very accurate information concerning the increase of the solutions of (6.411). We write (6.411) in the form $Qy' - P = 0$. The ratio of any two terms is of one of the forms

$$Ax^m y^n, \quad Ax^m y^n y',$$

where A is a constant, and is ultimately monotonic; and so, between any two terms X_i, X_j, there subsists one of the relations

$$X_i \succ X_j, \quad X_i \asymp X_j, \quad X_i \prec X_j.$$

It follows that there must be one pair of terms at any rate such that $X_i \asymp X_j$. If both or neither of X_i, X_j contain y', we obtain at once

(6.431) $$y \sim Ax^s,$$

where s is rational. If one only contains y', we obtain a relation

(6.432) $$y^m y' \sim Ax^n.$$

Here four cases present themselves. If $m \neq -1$, $n \neq -1$, we obtain a relation of the type (6.431). If $m \neq -1$, $n = -1$, we obtain

(6.433) $$y \sim A (\log x)^{1/p},$$

where p is an integer. If $m = -1$, $n \neq -1$, we obtain a relation

$$\log y \sim Ax^p,$$

(6.434) $$y = e^{Ax^p(1+\epsilon)}.$$

Here p may be supposed a positive integer, as $y \asymp 1$ if p is negative or zero*. Finally, if $m = -1$, $n = -1$, we obtain

$$\log y \sim A \log x,$$

(6.435) $$y = x^{A+\epsilon}.$$

This last form of y includes both (6.431) and (6.433) as special cases, since in the latter case $y = x^\epsilon$. We have thus proved

Theorem 35. *Any continuous solution of* (6.411) *is ultimately monotonic, and of one of the forms*

$$e^{Ax^p(1+\epsilon)}, \quad x^{A+\epsilon},$$

where p is a positive integer.

It is possible to go a good deal further. All derivatives of y are ultimately monotonic, and y satisfies one of the relations

$$y \sim Ax^a e^{\Pi(x)}, \quad y \sim A (x^p \log x)^{1/q},$$

where $\Pi(x)$ is a polynomial and p and q are integers.

For fuller developments see Hardy, **10**. For analogous investigations of equations of the second order, for which the possibilities are much more complex, see Fowler, **1**, **2**. These memoirs contain many additional references to the literature of the subject.

6.5. Oscillating Dirichlet's Integrals. The theory of Fourier series, when developed according to the ideas initiated by Dirichlet,

* p is clearly at most equal to $r+1$, where r is the degree of (6.411) in x: this, of course, agrees with Lindelöf's result quoted in § 6.41.

depends on Dirichlet's integral

$$J(\lambda)=\int_0^\xi \frac{\sin \lambda x}{x} f(x)\,dx \quad (\xi>0), \tag{6.51}$$

which has, under appropriate conditions, the limit $\frac{1}{2}\pi f(+0)$ when $\lambda \to \infty$.

A very interesting problem in the theory is that of finding asymptotic formulae for $J(\lambda)$ when

$$f(x)=\rho(x)\,e^{i\sigma(x)},$$

ρ and σ are L-functions, and $\sigma \to \infty$ when $x \to 0$. This problem was first attacked by du Bois-Reymond (**6**), who enunciated a number of striking theorems, but whose analysis is very inconclusive, and so obscure that it is almost impossible to distinguish between what he proved and what he did not*. The problem was reconsidered more recently by the author†, who obtained more definite results, and these results were afterwards completed in various respects by Kuniyeda‡.

In stating these results we assume throughout that $\rho \prec \sigma'$, this being the necessary and sufficient condition for the existence of the integral $J(\lambda)$. There are three cases which have to be distinguished, those in which

$$(A)\ \sigma \prec l\left(\frac{1}{x}\right), \quad (B)\ \sigma \asymp l\left(\frac{1}{x}\right), \quad (C)\ \sigma \succ l\left(\frac{1}{x}\right);$$

and the main theorems are as follows. The proofs are too elaborate for reproduction here.

Theorem 36. *If $\sigma \prec l(1/x)$ and $\rho = x^{-a}\Theta(x)$, where $x^{\delta} \prec \Theta \prec x^{-\delta}$, so that $a \leqq 1$, then*

$$J(\lambda)=O(\lambda^{-1+\delta}) \quad (a \leqq -1),$$

$$J(\lambda) \sim -\Gamma(-a)\sin\tfrac{1}{2}a\pi\,\rho\left(\frac{1}{\lambda}\right)e^{i\sigma(1/\lambda)} \quad (-1<a<1),$$

$$J(\lambda) \sim \lambda T\left(\frac{1}{\lambda}\right) \quad (a=1),$$

where

$$T(x)=\int_0^\lambda \rho(t)\,e^{i\sigma(t)}\,dt$$

and $-\Gamma(-a)\sin\frac{1}{2}a\pi$ is to be replaced by $\frac{1}{2}\pi$ when $a=0$.

Theorem 37. *If $\sigma \sim bl(1/x)$ then*

$$J(\lambda)=O(\lambda^{-1+\delta}) \quad (a \leqq -1),$$

$$J(\lambda) \sim -\Gamma(-a-bi)\sin\tfrac{1}{2}(a+bi)\pi\,\rho\left(\frac{1}{\lambda}\right)e^{i\sigma(1/\lambda)} \quad (-1<a\leqq 1).$$

* du Bois-Reymond asks only whether $J(\lambda)$ does or does not tend to a limit, and does not attempt to find asymptotic formulae in the case of oscillation.

† Hardy, **11**. ‡ Kuniyeda, **1**.

Theorem 38. *If* $l(1/x) \prec \sigma \prec (1/x)^{\Delta}$ *and* $\rho \prec x\sigma'$, *then*

$$J(\lambda) = O\left(\frac{1}{\lambda}\right)$$

if $\rho \preccurlyeq x\sqrt{\sigma''/\sigma'}$, *and*

$$J(\lambda) \sim \sqrt{(\tfrac{1}{2}\pi)}\, e^{(\beta - \frac{1}{4}\pi) i} \frac{\rho(\theta)}{\theta\sqrt{\{\sigma''(\theta)\}}}$$

if $x\sqrt{\sigma''/\sigma'} \prec \rho \prec x\sigma'$. *Here* $\beta = \lambda\theta + \sigma(\theta)$, *and* θ *is determined as a function of* λ *by* $\sigma'(\theta) + \lambda = 0$.

These theorems are stated in the form finally given to them by Kuniyeda. It should be observed that they are still not quite complete. No asymptotic formula has been obtained when $\sigma \succ (1/x)^{\Delta}$, and no account is taken, in Theorem 38, of the range $x\sigma' \preccurlyeq \rho \prec \sigma'$*. There is also room for a more accurate determination of the first formula of Theorems 36 and 37.

Kuniyeda has also investigated the integral $K(\lambda)$ in which $\cos \lambda x$ appears instead of $\sin \lambda x$, the results being of the same character. This integral appears in the theory of the trigonometrical series conjugate to the Fourier series of $f(x)$, and in the theory of power-series on the circle of convergence.

Apart from the work of du Bois-Reymond, special cases of the problem had already been considered by Darboux, Hamy, and Fejér†. In particular Fejér determined the asymptotic formula

$$a_n \sim \frac{1}{\sqrt{(e\pi)}} n^{-\frac{3}{4} + \frac{1}{2}p} \sin(2\sqrt{n} + \tfrac{3}{4}\pi - \tfrac{1}{2}p\pi)$$

for the coefficients in the power series

$$f(x) = (1-x)^{-p} e^{-1/(1-x)} = \Sigma a_n x^n.$$

6.61. Arithmetic applications. The classification of irrational numbers. We conclude with a brief sketch of some of the most important applications of the theory in arithmetical directions. These applications bear primarily on problems connected with the classification of irrational numbers.

An *algebraic number of degree k* is a root of an irreducible equation

$$f(x) = a_0 x^k + a_1 x^{k-1} + \ldots + a_k = 0, \qquad (6.611)$$

in which the coefficients are rational integers without common factor. If $a_0 = 1$, x is an *integer*. A number which is not algebraic is *transcendental*.

In what follows we confine our attention to real numbers. The aggregate of algebraic numbers is enumerable, and there are therefore transcendental numbers in every interval of the continuum‡. The

* See Kuniyeda, **1**, 35.

† Darboux, **1**; Hamy, **1**; Fejér, **1**, **2**.

‡ Cantor, **1**. For accounts of the relevant parts of Cantor's theory see Borel, **1**; Hausdorff, **1**; Hobson, **2**; Jourdain, **1**.

theory of aggregates establishes in this manner the existence of transcendental numbers, but does not suggest directly any method for constructing them. Such a construction was first effected by Liouville*, by means of the following theorem.

Theorem 39. *If x is an algebraic number of degree k, and p/q is a rational number not equal to x†, then there is a number $M = M(x)$, independent of q, such that*

(6.612) $$\left| x - \frac{p}{q} \right| > \frac{1}{Mq^k}.$$

Suppose, as plainly we may, that there is no better approximation to x, with denominator q, than p/q. Then p/q differs from x by less than $1/q$, and $|f'(y)|$ has, in the interval $|y - x| \leq 1/q$, an upper bound $\frac{1}{2}M$ independent of q‡. But

$$f\left(\frac{p}{q}\right) = f\left(\frac{p}{q}\right) - f(x) = \left(\frac{p}{q} - x\right) f'(y),$$

where y lies between x and p/q, and so

$$\left| x - \frac{p}{q} \right| > \frac{1}{M} \left| f\left(\frac{p}{q}\right) \right|.$$

As $|f(p/q)|$ is a rational number whose denominator is q^k and whose numerator is at least 1, the theorem follows. It is plain that

(6.613) $$\left| x - \frac{p}{q} \right| > \frac{1}{q^{k+1}}$$

for all sufficiently large values of q.

Liouville's theorem shows in effect that *it is impossible to approximate to an algebraic number by rationals with more than a certain accuracy.* On the other hand it is easy to write down particular irrationals which possess rational approximations of any degree of accuracy whatever. Suppose for example that ϕ_n is an increasing function of n, integral for every integral n, and let

$$x = 10^{-\phi_1} + 10^{-\phi_2} + \dots + 10^{-\phi_n} + \dots.$$

If p_n/q_n is the sum of the first n terms of the series, so that $q_n = 10^{-\phi_n}$, then

$$0 < x - \frac{p_n}{q_n} = 10^{-\phi_{n+1}} + 10^{-\phi_{n+2}} + \dots \leq \frac{10}{9} 10^{-\phi_{n+1}}$$

* Liouville, 2.

† This provision is naturally only necessary when $k = 1$.

‡ We have certainly, for example,

$$\tfrac{1}{2}M \leq k\,|a_0|\,(|x| + 1)^{k-1} + \dots + |a_{k-1}|.$$

and

(6.614) $$\left|x - \frac{p_n}{q_n}\right| \leq \frac{10}{9} q_n^{-\chi_n},$$

where $\chi_n = \phi_{n+1}/\phi_n$. If $\chi_n \to \infty$, (6.613) and (6.614) are contradictory, so that x is transcendental. We may for example take $\phi_n = n\,!$.

Cantor's theory shows that transcendental numbers exist, and Liouville's theorem enables us to produce examples of them. To prove that a particular number, arising independently in analysis, is transcendental, or even irrational, is in general a far more difficult problem. It has never been proved, for example, that $2^{\sqrt{2}}$, e^{π}, or Euler's constant γ are irrational.

There are a few classes of numbers, such as $\sqrt{2}$, $\sqrt{n}$, $\sqrt[3]{2}$, e, $\log_{10} 2$, ..., whose irrationality is classical: see for example Hardy, **1**, 6, 380, 387. For the irrationality of π, first proved by Lambert (**1**), and π^2, see Perron, **1**, 254; Vahlen, **1**, 319. The problem of proving that $\sqrt[3]{2}$ is not expressible by any finite combination of quadratic surds is famous historically: see Enriques, **1**; Hudson, **1**; Klein, **1**. For an elementary proof that e is not quadratic, see Vahlen, **1**, 325. The transcendentality of e was first proved by Hermite (**1**), and that of π by Lindemann (**1**); full accounts of these problems are given by Enriques and Klein, and also by Hessenberg, **1**; Hobson, **3**; Perron, **2**. See also Maillet, **1**.

6.62. In the preceding construction, there is naturally no special merit in the number 10. We may use any other scale; and we may also employ other representations of irrationals, for example by continued fractions. The number

$$x = \frac{1}{a_1+}\,\frac{1}{a_2+}\,\cdots$$

will certainly be transcendental if a_n increases with sufficient rapidity, for, if p_n/q_n is the nth convergent, a_n' the complete quotient corresponding to a_n, and $q_n' = a_n' q_{n-1} + q_{n-2}$, we have

$$\left|x - \frac{p_n}{q_n}\right| = \frac{1}{q_n q'_{n+1}} < \frac{1}{a_{n+1} q_n^2};$$

and, in order to obtain a contradiction with (6.613), it is only necessary to suppose that $a_{n+1} \succ q_n^{\Delta}$ or, what is equivalent, that $q_{n+1} \succ q_n^{\Delta}$. It is easily proved that this is so whenever $a_{n+1} \succ a_n^{\Delta}$. Thus we might take

$$a_1 = 1,\ a_2 = 2^{a_1} = 2,\ \ldots,\ a_{n+1} = 2^{a_n},\ \ldots.$$

When $k=1$ or $k=2$, Liouville's theorem is, in a sense, final: it is not possible to replace the q^k on the right hand side by any lower power of q. When $k > 2$, more is true: thus Thue (**1**) proved that

$$\left|x - \frac{p}{q}\right| > \frac{1}{Mq^{\frac{1}{2}k+1+\epsilon}},$$

where $M = M(x, \epsilon)$, and Siegel (1) that

$$\left|x - \frac{p}{q}\right| > \frac{1}{Mq^{2\sqrt{k}}},$$

where $M = M(x)$. The index assigned by Siegel's theorem is better if $k > 11$. The problem of finding the best possible index is unsolved, except when k is 1 or 2.

When $k = 2$, the continued fraction for x is periodic, so that $a_n = O(1)$. It is natural to ask whether anything can be said about the order of a_n when x is an algebraic number of higher degree. It is easy to deduce from Liouville's theorem that $a_n \prec e_2(an)$, where a is a number depending on k, and similar deductions can be drawn from Thue's and Siegel's theorems. What can be proved in this way amounts to very little, and it is very unlikely that it is anywhere near the ultimate truth.

6.63. Although so little is known about the order of magnitude of a_n for particular classes of irrationals, very interesting results have been found concerning what may be called its 'usual' order of magnitude. We may say that x has *usually* the property P, or that P is usually true, if the set of values of x for which P is false has measure zero. If then ϕ_n is an increasing function of n, and we write $\phi_n = k_n$ or $\phi_n = d_n$, according as $\Sigma(1/\phi_n)$ is convergent or divergent, then

$$a_n < k_n \qquad (n > n_0)$$

is usually true, and

$$a_n < d_n \qquad (n > n_0)$$

is usually false*. Thus $a_n < n(ln)^2$ is usually true, and $a_n < n\, ln$ is usually false†.

It is easily proved that, if

$$\left|x - \frac{p}{q}\right| < \frac{1}{q\phi_q} \tag{6.631}$$

for an infinity of values of q, and $\phi_q = k_q$, then $a_n > k_n$ for an infinity of values of n‡. Hence (6.631) is in this case usually false.

We may ask generally for what irrationals (6.631) is infinitely often true. The results known in this direction are as follows. If ϕ_q is a constant C, and $C \leq \sqrt{5}$, then (6.631) is *always* true (for an infinity of values of q). If

$$\sqrt{5} < C \leq 2\sqrt{2},$$

then (6.631) is true except for irrationals equivalent§ to

$$a = \frac{1}{1+}\,\frac{1}{1+}\cdots.$$

If $2\sqrt{2} < C < 3$, then (6.631) is true except for the numbers equivalent to one or other of a finite number of quadratic surds. If $C \geq 3$, it is usually true, but the exceptions are non-enumerable. It is still usually true if ϕ_q is an increasing function whose increase is sufficiently slow; but it is usually false when

* That is to say, it is usually true that $a_n > d_n$ for an infinity of values of n.

† Borel, 9; Bernstein, 1.

‡ Here k_n is some function of n such that $\Sigma(1/k_n)$ is convergent. It is not the same function of n that k_q is of q.

§ *I.e.* numbers $(aa+b)/(ca+d)$, where a, b, c, d are integers and $ad - bc = 1$.

$\phi_q = k_q$. For fuller information see Borel, **5**; Bohr and Cramér, **1**; Grace, **1**; Heawood, **1**; Hermite, **1**; Hurwitz, **1**; Markoff, **1**; Minkowski, **1**; Perron, **2**.

Yet another closely allied problem is that of the distribution of the numbers (nx), where x is irrational, and $(u) = u - [u]$, in the interval (0, 1). The fundamental theorem, due to Kronecker, is that the numbers lie everywhere dense in the interval. There are many memoirs concerned with this theorem and its extensions. See Behnke, **1**; Bohr, **1**; Bohr and Cramér, **1**; Hardy and Littlewood, **3**; Hecke, **1**; Kronecker, **1**, **2**; Lettenmeyer, **1**; Minkowski, **1**; Ostrowski, **1**; Weyl, **1**.

6.64. Applications to the theory of convergence. Liouville's theorem, and the other theorems of which we have spoken, have many interesting applications to the theory of convergence of series.

The typical problem is that of the convergence of the series

$$\Sigma \frac{\phi_n}{|\sin n\pi x|}, \tag{6.641}$$

where ϕ_n is a decreasing function of n and x is irrational. If p_ν/q_ν is a convergent to x, then

$$|\sin q_\nu \pi x| < \frac{A}{q_{\nu+1}} < \frac{A}{a_{\nu+1} q_\nu},$$

where the A's are constants, and the increase of $a_{\nu+1}$, regarded as a function of q_ν, may be as rapid as we please. It follows that (6.641) *is divergent, for appropriate values of x, however rapid the decrease of ϕ_n may be.*

If x is an algebraic number of degree k, then, by (6.612),

$$|\sin n\pi x| > \frac{B}{n^{k-1}},$$

where B is a positive function of x only, for all values of n. Hence (6.641) is convergent whenever $\phi_n < n^{-a}$ and $a > k$: this result can naturally be improved upon by the use of Thue's and Siegel's theorems. Thus

$$\Sigma \frac{n^{-2-\delta}}{|\sin n\pi x|}$$

is convergent for all quadratic x, and

$$\Sigma \frac{e^{-\delta n}}{|\sin n\pi x|}$$

is convergent for all algebraic x. The 2 in the first of these results may in fact be replaced by 1, but a more elaborate proof is needed. It also follows from the results of § 6.63 that (6.641) is usually convergent if $\Sigma k_q \phi_q$ is convergent, as for example if $\phi_q = q^{-2} (\log q)^{-4}$, when we may take $k_q = q (\log q)^2$.

The series $\Sigma z^n \operatorname{cosec} n\pi x$ may, according to the arithmetic nature of x, represent an integral function of z, or a function regular inside a circle which is a line of singularities of the function; or again it may diverge for all values of z.

The theory of the non-absolute convergence of such a series as $\Sigma \phi_n \operatorname{cosec} n\pi x$ is naturally more intricate.

For fuller information see Hardy, **5**; Hardy and Littlewood, **3** (3); Lerch, **1**; Riemann, **2**; Smith, **1**. Analogous questions concerning integrals are discussed by Hardy, **3** (5).

APPENDIX

SOME NUMERICAL ILLUSTRATIONS*

1. *Table of the functions log x, log log x, log log log x, etc.*

x	$\log x$	$\log_2 x$	$\log_3 x$	$\log_4 x$	$\log_5 x$
10	2·30	0·834	−0·182	—	—
10^3	6·91	1·933	0·659	−0·417	—
10^6	13·82	2·626	0·966	−0·035	—
10^{10}	23·03	3·137	1·143	0·134	−2·011
10^{15}	34·54	3·542	1·265	0·235	−1·449
10^{20}	46·05	3·830	1·343	0·295	−1·221
10^{30}	69·08	4·235	1·443	0·367	−1·003
10^{60}	138·15	4·928	1·595	0·467	−0·762
10^{100}	230·26	5·439	1·693	0·527	−0·641
10^{1000}	2302·58	7·742	2·047	0·716	−0·334
10^{10^6}	2303×10^3	14·650	2·685	0·987	−0·013
$10^{10^{10}}$	2303×10^7	23·860	3·172	1·154	0·144

2. *Table of the functions e^x, e^{e^x}, $e^{e^{e^x}}$, etc.*

x	ex	$e_2 x$	$e_3 x$	$e_4 x$
1	2·718	15·154	3,814,260	$10^{1,656,510}$
2	7·389	1618·2	$5·85 \times 10^{702}$	—
3	20·085	$5·28 \times 10^8$	$10^{2·295 \times 10^8}$	—
5	148·413	$2·85 \times 10^{64}$	$10^{1·24 \times 10^{64}}$	—
10	22026	$9·44 \times 10^{9565}$	—	—

The function $\log x$ is defined only for $x > 0$, $\log_2 x$ for $x > 1$, $\log_3 x$ for $x > e$, $\log_4 x$ for $x > e^e = e_2$, and so on. The values of the first few numbers e, e_2, e_3, ... are given above, viz. $e = 2·718$, $e_2 = 15·154$, $e_3 = 3,814,260$, $e_4 = 10^{1,656,510}$.

* The tables in this appendix were calculated by Mr J. Jackson.

3. *Table to illustrate the convergence of the series*

(1) $\sum_{3}^{\infty} \frac{1}{n\,ln\,(lln)^2}$. (2) $\sum_{2}^{\infty} \frac{1}{n\,(ln)^2}$. (3) $\sum_{1}^{\infty} \frac{1}{n^s}$. (4) $\sum_{0}^{\infty} x^n$.

(5) $\sum_{0}^{\infty} \frac{1}{n!}$. (6) $\sum_{1}^{\infty} \frac{1}{n^n}$ (7) $\sum_{0}^{\infty} x^{n^2}$. (8) $\sum_{1}^{\infty} n^{-n^n}$.

Series	Sum	Number of terms required to calculate the sum correctly to 2 decimal places	10	100	1000
1	38·43	$10^{3\cdot 14\times 10^{86}}$	—	—	—
2	2·11	$7\cdot 23\times 10^{86}$	$10^{8\cdot 6\times 10^{9}}$	—	—
3 (s = 1·1)	10·58	10^{33}	10^{113}	10^{1013}	10^{10013}
3 (s = 1·5)	2·612	160,000	16×10^{20}	16×10^{200}	16×10^{2000}
3 (s = 2)	$\frac{1}{6}\pi^2 = 1\cdot 64493$	200	2×10^{10}	2×10^{100}	2×10^{1000}
3 (s = 10)	1·0009846	1	11	$1\cdot 093\times 10^{11}$	$1\cdot 093\times 10^{111}$
3 (s = 100)	$1+(1\cdot 27\times 10^{-30})$	1	1	10	$1\cdot 213\times 10^{10}$
4 (x = ·9)	10	73	247	2214	21883
4 (x = ·3)	2	9	36	336	3325
4 (x = ·1)	10/9	3	11	101	1001
5	$e-1 = 1\cdot 718282$	5	13	70	440
6	1·291286	3	10	57	386
7 (x = ·9)	3·234989	8	15	46	148
7 (x = ·5)	1·564468	3	6	19	58
7 (x = ·1)	1·100100	2	4	11	32
8	1·062500	2	2	3	4

The phrase 'calculate the sum correctly to m decimal places' is used as equivalent to 'calculate with an error less than $\frac{1}{2}\times 10^{-m}$'. In the case of a very slowly convergent series the interpretation affects the numbers to a considerable extent. The numbers would be considerably more difficult to calculate were the phrase interpreted in its literal sense.

Such a series as 3 (s = 100) is of course exceedingly rapidly convergent *at first*, *i.e.* a very few terms suffice to give the sum correctly to a considerable number of places; but if the sums are wanted to a very large number of places, even the series 4 (x = ·9) proves to be far more practicable.

4. *Table to illustrate the divergence of the series*

(1) $\dfrac{1}{\log\log 3}+\dfrac{1}{\log\log 4}+\dots$ (2) $\dfrac{1}{\log 2}+\dfrac{1}{\log 3}+\dots$

(3) $1+\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}+\dots$ (4) $1+\frac{1}{2}+\frac{1}{3}+\dots$

(5) $\dfrac{1}{2\log 2}+\dfrac{1}{3\log 3}+\dots$ (6) $\dfrac{1}{3\log 3\log\log 3}+\dfrac{1}{4\log 4\log\log 4}$.

Series	Number of terms required to make the sum greater than					
	3	5	10	100	1000	10^6
1	1	1	1	116	1800	$2{\cdot}6\times 10^6$
2	3	7	20	440	7600	$1{\cdot}5\times 10^7$
3	5	10	33	2500	$2{\cdot}5\times 10^5$	$2{\cdot}5\times 10^{11}$
4	11	82	12390	10^{43}	$10^{{\cdot}43\times 10^3}$	$10^{{\cdot}43\times 10^6}$
5	8690	$1{\cdot}3\times 10^{29}$	10^{4300}	$10^{5\times 10^{42}}$	——	——
6	1	60 *to* 70	$10^{10^{100}}$	——	——	——

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The General Theory of Dirichlet's Series

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THE GENERAL THEORY OF DIRICHLET'S SERIES

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PREFACE

THE publication of this tract has been delayed by a variety of causes, and I am now compelled to issue it without Dr Riesz's help in the final correction of the proofs. This has at any rate one advantage, that it gives me the opportunity of saying how conscious I am that whatever value it possesses is due mainly to his contributions to it, and in particular to the fact that it contains the first systematic account of his beautiful theory of the summation of series by 'typical means'.

The task of condensing any account of so extensive a theory into the compass of one of these tracts has proved an exceedingly difficult one. Many important theorems are stated without proof, and many details are left to the reader. I believe, however, that our account is full enough to serve as a guide to other mathematicians researching in this and allied subjects. Such readers will be familiar with Landau's *Handbuch der Lehre von der Verteilung der Primzahlen*, and will hardly need to be told how much we, in common with all other investigators in this field, owe to the writings and to the personal encouragement of its author.

G. H. H.

19 *May* 1915.

CONTENTS

THE GENERAL THEORY OF DIRICHLET'S SERIES

I

INTRODUCTION

1. The series whose theory forms the subject of this tract are of the form

$$f(s)=\sum_{1}^{\infty} a_n e^{-\lambda_n s} \quad \text{..........................(1)},$$

where (λ_n) is a sequence of real increasing numbers whose limit is infinity, and $s=\sigma+ti$ is a complex variable whose real and imaginary parts are σ and t. Such a series is called a Dirichlet's series of *type* λ_n. If $\lambda_n=n$, then $f(s)$ is a power series in e^{-s}. If $\lambda_n=\log n$, then

$$f(s)=\sum_{1}^{\infty} a_n n^{-s} \quad \text{............................(2)}$$

is called an *ordinary* Dirichlet's series.

Dirichlet's series were, as their name implies, first introduced into analysis by Dirichlet, primarily with a view to applications in the theory of numbers. A number of important theorems concerning them were proved by Dedekind, and incorporated by him in his later editions of Dirichlet's *Vorlesungen über Zahlentheorie*. Dirichlet and Dedekind, however, considered only real values of the variable s. The first theorems involving complex values of s are due to Jensen*, who determined the nature of the region of convergence of the general series (1); and the first attempt to construct a systematic theory of the function $f(s)$ was made by Cahen† in a memoir which, although much of the analysis which it contains is open to serious criticism, has

* Jensen, **1**, **2**. References in thick type are to the bibliography at the end of the tract.
† Cahen, **1**.

served—and possibly just for that reason—as the starting point of most of the later researches in the subject*.

It is clear that all but a finite number of the numbers λ_n must be positive. It is often convenient to suppose that they are all positive, or at any rate that $\lambda_1 \geqq 0$.†

2. It will be convenient at this point to fix certain notations which we shall regard as stereotyped throughout the tract.

(i) By $[x]$ we mean the algebraically greatest integer not greater than x. By

$$\sum_{\alpha}^{\beta} f(n)$$

we mean the sum of all values of $f(n)$ for which $\alpha \leqq n \leqq \beta$, *i.e.* for $[\alpha] \leqq n \leqq [\beta]$ or $[\alpha] < n \leqq [\beta]$, according as α is or is not an integer. We shall also write

$$A(x) = \sum_{1}^{x} a_n, \quad A(x, y) = \sum_{x}^{y} a_n, \ddagger$$

$$\Delta a_n = a_n - a_{n+1}.$$

(ii) We shall follow Landau in his use of the symbols o, O.§ That is to say, if ϕ is a positive function of a variable which tends to a limit, we shall write

$$f = o(\phi)$$

if $f/\phi \to 0$, and

$$f = O(\phi)$$

if $|f|/\phi$ remains less than a constant K. We shall use the letter K to denote an unspecified constant, not always the same||.

* Fuller information as to the history of the subject (up to 1909) will be found in Landau's *Handbuch der Lehre von der Verteilung der Primzahlen*, Vol. 2, Book 6, Notes and Bibliography, and in the *Encycl. des sc. math.*, T. 1, Vol. 3, pp. 249 *et seq.* We shall refer to Landau's book by the letter *H*. The two volumes are paged consecutively.

† It is evident that we can reduce the series (1) to a series satisfying this condition either (*a*) by subtracting from $f(s)$ a finite sum $\Sigma a_n e^{-\lambda_n s}$ or (*b*) by multiplying $f(s)$ by an exponential e^{-Cs}. These operations would of course change the type of the series.

‡ We shall use the corresponding notations, with letters other than a, without further explanation.

§ Landau, *H.*, p. 883, states that the symbol O seems to have been first used by Bachmann, *Analytische Zahlentheorie*, Vol. 2, p. 401.

|| For fuller explanations see Hardy, *Orders of infinity* (Camb. Math. Tracts, No. 12), pp. 5 *et seq.*

II

ELEMENTARY THEORY OF THE CONVERGENCE OF DIRICHLET'S SERIES

1. Two fundamental lemmas. Much of our argument will be based upon the two lemmas which follow.

LEMMA 1. *We have identically*

$$\sum_x^y a_\nu \phi(\nu) = \sum_x^{y-1} A(x, n) \Delta \phi(n) + A(x, y) \phi[y].$$

This is Abel's classical lemma on partial summation *.

LEMMA 2. *If* $\sigma \neq 0$, *then*

$$|\Delta e^{-\lambda_n s}| \leqq \frac{|s|}{\sigma} \Delta e^{-\lambda_n \sigma}. \dagger$$

For

$$|\Delta e^{-\lambda_n s}| = \left| \int_{\lambda_n}^{\lambda_{n+1}} s e^{-us} \, du \right| \leqq |s| \int_{\lambda_n}^{\lambda_{n+1}} e^{-u\sigma} \, du = \frac{|s|}{\sigma} \Delta e^{-\lambda_n \sigma}.$$

2. Fundamental Theorems. Region of convergence, analytical character, and uniqueness of the series. We are now in a position to prove the most important theorems in the elementary theory of Dirichlet's series.

THEOREM 1. *If the series is convergent for* $s = \sigma + ti$, *then it is convergent for any value of s whose real part is greater than* σ.

This theorem is included in the more general and less elementary theorem which follows.

THEOREM 2. *If the series is convergent for* $s = s_0$, *then it is uniformly convergent throughout the angular region in the plane of s defined by the inequality*

$$|\text{am}(s - s_0)| \leqq \alpha < \tfrac{1}{2}\pi. \ddagger$$

* Abel, **1**.

† This lemma seems to have been stated first in this form by Perron, **1**, but is contained implicitly in many earlier writings.

‡ If $s = re^{i\theta}$, we write $r = |s|$, $\theta = \text{am}\, s$. Theorem 1 is due to Jensen, **1**, and Theorem 2 to Cahen, **1**.

We may clearly suppose $s_0 = 0$ without loss of generality. We have

$$\sum_m^n a_\nu e^{-\lambda_\nu s} = \sum_m^{n-1} A(m, \nu) \Delta e^{-\lambda_\nu s} + A(m, n) e^{-\lambda_n s},$$

by Lemma 1. If ϵ is assigned we can choose m_0 so that $\lambda_m > 0$ and

$$|A(m, \nu)| < \epsilon \cos \alpha$$

for $\nu \geqq m \geqq m_0$. If now we apply Lemma 2, and observe that

$$|s|/\sigma \leqq \sec \alpha$$

throughout the region which we are considering, we obtain

$$\left| \sum_m^n a_\nu e^{-\lambda_\nu s} \right| < \epsilon \left(\sum_m^{n-1} \Delta e^{-\lambda_\nu \sigma} + e^{-\lambda_n \sigma} \right) = \epsilon e^{-\lambda_m \sigma} < \epsilon$$

for $n \geqq m \geqq m_0$. Thus Theorem 2 is proved*, and Theorem 1 is an obvious corollary.

There are now three possibilities as regards the convergence of the series. It may converge for *all*, or *no*, or *some* values of s. In the last case it follows from Theorem 1, by a classical argument, that we can find a number σ_0 such that the series is convergent for $\sigma > \sigma_0$ and divergent or oscillatory for $\sigma < \sigma_0$.

THEOREM 3. *The series may be convergent for all values of s, or for none, or for some only. In the last case there is a number σ_0 such that the series is convergent for $\sigma > \sigma_0$ and divergent or oscillatory for $\sigma < \sigma_0$.*

In other words *the region of convergence is a half-plane*†. We shall call σ_0 the *abscissa of convergence*, and the line $\sigma = \sigma_0$ the *line of convergence*. It is convenient to write $\sigma_0 = -\infty$ or $\sigma_0 = \infty$ when the series is convergent for all or no values of s. *On* the line of convergence the question of the convergence of the series remains open, and requires considerations of a much more delicate character.

* It is possible to substitute for the angle considered in this theorem a wider region; *e.g.* the region

$$\sigma \geqq 0, \quad |t| \leqq e^{K\sigma} - 1$$

(Perron, **1**; Landau, *H.*, p. 739). We shall not require any wider theorem than 2. It may be added that the result of Theorem 1 remains true when we only assume that Σa_n is at most finitely oscillating: in fact, with this hypothesis, the result of Theorem 2 holds for the region

$$|\text{am}(s - s_0)| \leqq \alpha < \tfrac{1}{2}\pi, \quad \sigma \geqq \delta > 0,$$

as is easily proved by a trifling modification of the argument given above.

† Jensen, **1**.

3. Examples. (i) The series $\Sigma a^n n^{-s}$, where $|a| < 1$, is convergent for all values of s.

(ii) The series $\Sigma a^n n^{-s}$, where $|a| > 1$, is convergent for no values of s.

(iii) The series Σn^{-s} has $\sigma = 1$ as its line of convergence. It is not convergent at any point of the line of convergence, diverging to $+\infty$ for $s = 1$, and oscillating finitely* at all other points of the line.

(iv) The series $\sum_{2}^{\infty} (\log n)^{-2} n^{-s}$ has the same line of convergence as the last series, but is convergent (indeed absolutely convergent) at all points of the line.

(v) The series $\sum_{2}^{\infty} a_n n^{-s}$, where $a_n = (-1)^n + (\log n)^{-2}$, has the same line of convergence, and is convergent (though not absolutely) at all points of it†.

4. THEOREM 4. *Let D denote any finite region in the plane of s for all points of which*

$$\sigma \geqq \sigma_0 + \delta > \sigma_0.$$

Then the series is uniformly convergent throughout D, and its sum $f(s)$ is a branch of an analytic function, regular throughout D. Further, the series

$$\Sigma a_n \lambda_n^{\rho} e^{-\lambda_n s},$$

where ρ is any number real or complex, and λ_n^{ρ} has its principal value, is also uniformly convergent in D, and, when ρ is a positive integer, represents the function

$$(-1)^{\rho} f^{(\rho)}(s).$$

The uniform convergence of the original series follows at once from Theorem 2, since we can draw an angle of the type considered in that theorem and including D‡. The remaining results, in so far as they concern the original series and its derived series, then follow immediately from classical theorems of Weierstrass§.

When ρ is not a positive integer, we choose a positive integer m so that the real part of $\rho - m$ is negative. The series

$$\Sigma a_n \lambda_n^{\rho - m} e^{-\lambda_n s} \quad \text{.........................} \quad (1)$$

may be written in the form

$$\Sigma b_n e^{-(m-\rho)\log \lambda_n} \quad \text{.........................} \quad (2),$$

where $b_n = a_n e^{-\lambda_n s}$. Regarding (2) as a Dirichlet's series of type $\log \lambda_n$, and applying Theorem 1, we see that (1) is convergent whenever

* See, *e.g.*, Bromwich, 2.

† We are indebted to Dr Bohr for this example.

‡ The vertex of the angle may be taken at σ_0, if the series is convergent for $s = \sigma_0$, and otherwise at $\sigma_0 + \eta$, where $0 < \eta < \delta$.

§ See, *e.g.*, Weierstrass, *Abhandlungen aus der Funktionentheorie*, pp. 72 *et seq.*; Osgood, *Funktionentheorie*, Vol. 1, pp. 257 *et seq.*

$\Sigma a_n e^{-\lambda_n s}$ is convergent. The proof of the theorem may now be completed by a repetition of our previous arguments.

THEOREM 5. *If the series is convergent for $s = s_0$, and has the sum $f(s_0)$, then $f(s) \to f(s_0)$ when $s \to s_0$ along any path which lies entirely within the region*

$$|\operatorname{am}(s - s_0)| \leqq \alpha < \tfrac{1}{2}\pi.$$

This theorem* is an immediate corollary from Theorem 2. It is of course only when s_0 lies on the line of convergence that it gives us any information beyond what is given by Theorem 4.

5. THEOREM 6. *Suppose that the series is convergent for $s = 0$, and let E denote the region*

$$\sigma \geqq \delta > 0, \quad |\operatorname{am} s| \leqq \alpha < \tfrac{1}{2}\pi.$$

Suppose further that $f(s) = 0$ for an infinity of values of s lying inside E. Then $a_n = 0$ for all values of n.

The function $f(s)$ cannot have an infinity of zeros in the neighbourhood of any finite point of E, since it is regular at any such point. Hence we can find an infinity of values $s_n = \sigma_n + t_n i$, where $\sigma_{n+1} > \sigma_n$, $\lim \sigma_n = \infty$, such that $f(s_n) = 0$.

But $$g(s) = e^{\lambda_1 s} f(s) = a_1 + \sum_2^{\infty} a_n e^{-(\lambda_n - \lambda_1)s}$$

is convergent for $s = 0$ and so uniformly convergent in E. Hence

$$g(s) \to a_1$$

when $s \to \infty$ along any path in E. This contradicts the fact that $g(s_n) = 0$, unless $a_1 = 0$. It is evident that we may repeat this argument and so complete the proof of the theorem†.

6. Determination of the abscissa of convergence. Let us suppose that the series is not convergent for $s = 0$, and let

$$\overline{\lim} \frac{\log |A(n)|}{\lambda_n} = \gamma. \ddagger$$

* The generalisation of the 'Abel-Stolz' theorem for power series (Abel, **1**; Stolz, **1**, **2**).

† This theorem, like Theorem 2 itself, may be made wider: see Perron, **1**; Landau, *H.*, p. 745. Until recently it was an open question whether it were possible that $f(s)$ could have zeros whose real parts surpass all limit: all that Theorem 6 and its generalisations assert is that the imaginary parts of such zeros, if they exist, must increase with more than a certain rapidity. The question has however been answered affirmatively by Bohr, **4**. But if there is a region of absolute convergence, the answer is negative (see III, § 5).

‡ By $\overline{\lim}\, u_n$ we denote the 'maximum limit' of the sequence u_n: *cf.* Bromwich, *Infinite series*, p. 13.

It is evident that $\gamma \geqq 0$*. We shall now prove that $\sigma_0 = \gamma$.

(i) Let δ be any positive number. We shall prove first that the series is convergent for $s = \gamma + \delta$.

Choose ϵ so that $0 < \epsilon < \delta$. Then, by the definition of γ, we have

$$\log |A(\nu)| < (\gamma + \delta - \epsilon)\lambda_\nu, \quad |A(\nu)| < e^{(\gamma+\delta-\epsilon)\lambda_\nu}$$

for sufficiently large values of ν. Now

$$\sum_1^n a_\nu e^{-\lambda_\nu s} = \sum_1^{n-1} A(\nu)\,\Delta e^{-\lambda_\nu s} + A(n)\, e^{-\lambda_n s}.$$

The last term is, for sufficiently large values of n, less in absolute value than $e^{-\epsilon\lambda_n}$, and so tends to zero; and everything depends on establishing the convergence of the series

$$\sum e^{(\gamma+\delta-\epsilon)\lambda_\nu}\,\Delta e^{-(\gamma+\delta)\lambda_\nu}.$$

Now, since $\gamma + \delta - \epsilon$ is positive, we have

$$e^{(\gamma+\delta-\epsilon)\lambda_\nu}\Delta e^{-(\gamma+\delta)\lambda_\nu} = (\gamma+\delta)\int_{\lambda_\nu}^{\lambda_{\nu+1}} e^{(\gamma+\delta-\epsilon)\lambda_\nu - (\gamma+\delta)x}\, dx$$

$$< (\gamma+\delta)\int_{\lambda_\nu}^{\lambda_{\nu+1}} e^{-\epsilon x}\, dx;$$

and the series $$(\gamma+\delta)\sum \int_{\lambda_\nu}^{\lambda_{\nu+1}} e^{-\epsilon x}\, dx$$

is obviously convergent. It follows that

$$\sigma_0 \leqq \gamma.$$

(ii) Suppose $$\sum a_\nu e^{-\lambda_\nu s} = \sum b_\nu \qquad (s > 0)$$
convergent. Then

$$A(n) = \sum_1^n b_\nu e^{\lambda_\nu s} = \sum_1^{n-1} B(\nu)\,\Delta e^{\lambda_\nu s} + B(n)\, e^{\lambda_n s}.$$

It follows that $$|A(n)| < K e^{\lambda_n s},$$

and therefore that

$$\log |A(n)| < \lambda_n s + K < (s+\delta)\lambda_n,$$

for any positive δ, if n is large enough. Hence

$$s \geqq \overline{\lim} \frac{\log |A(n)|}{\lambda_n} = \gamma,$$

and therefore $$\sigma_0 \geqq \gamma.$$

* We can determine a constant K such that $\log |A(n)| > -K$ for an infinity of values of n. This would still be true if $\sum a_n$ converged to a sum other than zero: but if the sum were zero we should have

$$\log |A(n)| \to -\infty.$$

From the results of (i) and (ii) we deduce

THEOREM 7. *If the abscissa of convergence of the series is positive, it is given by the formula*

$$\sigma_0 = \overline{\lim} \frac{\log |A(n)|}{\lambda_n}.*$$

7. Absolute convergence of Dirichlet's series. We can apply the arguments of the preceding sections to the series

$$\Sigma |a_n| e^{-\lambda_n s} \qquad \text{.............................(1).}$$

We deduce the following result:

THEOREM 8. *There is a number $\bar{\sigma}$ such that the series* (1) *is absolutely convergent if $\sigma > \bar{\sigma}$ and is not absolutely convergent if $\sigma < \bar{\sigma}$. This number, if positive, is given by the formula*

$$\bar{\sigma} = \overline{\lim} \frac{\log \bar{A}(n)}{\lambda_n},$$

where $\bar{A}(n) = |a_1| + |a_2| + \ldots + |a_n|.$

In other words a Dirichlet's series possesses, besides its abscissa, line, and half-plane of convergence, an abscissa, line, and half-plane of absolute convergence. It should however be observed that the theorem which asserts the existence of a half-plane of absolute convergence is in reality more elementary than Theorem 3, as it follows at once from the inequality

$$|e^{-\lambda_\nu s}| \leqq |e^{-\lambda_\nu s_1}| \qquad (\sigma \geqq \sigma_1),$$

and does not depend on Lemma 1.

It is evident that $\bar{\sigma} \geqq \sigma_0$. We may of course have $\bar{\sigma} = \infty$ or $\bar{\sigma} = -\infty$. In general there will be a *strip* between the lines of convergence and absolute convergence, throughout which the series is conditionally convergent. This strip may vanish (if $\bar{\sigma} = \sigma_0$) or comprise the whole plane (if $\sigma_0 = -\infty$, $\bar{\sigma} = \infty$) or a half-plane (if

* Cahen, **1**. Dedekind, *l.c.* p. 1, and Jensen, **2**, had already given results which together contain the substance of the theorem. The result holds when $\sigma_0 = 0$, unless Σa_n converges to zero. If $\sigma_0 < 0$ the result is in general untrue. It is plain that in such a case we can find σ_0 by first applying to the variable s such a linear transformation as will make the abscissa of convergence positive. But there is a formula directly applicable to this case, viz.

$$\sigma_0 = \overline{\lim} \frac{\log |A - A(n)|}{\lambda_{n+1}},$$

where A is the sum of the series Σa_n (obviously convergent when $\sigma_0 < 0$). This formula was given (with a slight error, viz. λ_n for λ_{n+1}) by Pincherle, **1**: see also Knopp, **6**; Schnee, **6**. Formulae applicable in *all* cases have been found by Knopp, **6** (for the case $\lambda_n = \log n$ only); Kojima, **1**; Fujiwara, **1**; and Lindh (Mittag-Leffler, **1**).

$\sigma_0 = -\infty,\ -\infty < \bar{\sigma} < \infty$ or $-\infty < \sigma_0 < \infty,\ \bar{\sigma} = \infty$). For Dirichlet's series of a given type, however, its breadth is subject to a certain limitation.

THEOREM 9. *We have* $\bar{\sigma} - \sigma_0 \leqq \overline{\lim} \dfrac{\log n}{\lambda_n}$.

We shall prove this theorem on the assumption that $\sigma_0 > 0$; its truth is obviously independent of this restriction. Given δ, we can choose n_0 so that

$$|A(n)| < e^{(\sigma_0+\delta)\lambda_n} \qquad (n \geqq n_0),$$

and accordingly

$$|a_n| = |A(n) - A(n-1)| < 2e^{(\sigma_0+\delta)\lambda_n} < e^{(\sigma_0+2\delta)\lambda_n}.*$$

Hence
$$\bar{A}(n) = \sum_1^n |a_\nu| < \bar{A}(n_0) + ne^{(\sigma_0+2\delta)\lambda_n} < ne^{(\sigma_0+3\delta)\lambda_n}$$

if $n \geqq n_1$ and n_1 is sufficiently large in comparison with n_0. Thus

$$\frac{\log \bar{A}(n)}{\lambda_n} < \frac{\log n}{\lambda_n} + \sigma_0 + 3\delta \qquad (n \geqq n_1),$$

from which the theorem follows immediately.

If $\log n = o(\lambda_n)$, the lines of convergence and absolute convergence coincide: in particular this is the case if $\lambda_n = n$. In this case our theorems become, on effecting the transformation $e^{-s} = x$, classical theorems in the theory of power series. Thus Theorems 1 and 3 establish the existence of the circle of convergence, and 7 gives a slightly modified form of Cauchy's formula for the radius of convergence. Theorems 2, 4, 5, and 6 also become familiar results. If $\lambda_n = \log n$, the maximum possible distance between the lines of convergence is 1. This is of course an obvious consequence of the fact that $\sum n^{-1-\delta}$ is convergent for all positive values of δ.

It is not difficult to construct examples to show that every logically possible disposition of the lines of convergence and absolute convergence, consistent with Theorem 9, may actually occur. We content ourselves with mentioning the series

$$\sum \frac{(-1)^n}{\sqrt{n}} (\log n)^{-s},$$

which is convergent for all values of s, but never absolutely convergent.

8. It will be well at this point to call attention to the essential difference which distinguishes the general theory of Dirichlet's series from the simpler theory of power series, and lies at the root of the particular difficulties of the former. The region of convergence of a power series is determined in the simplest possible manner by the disposition of the singular points of the function which it represents: the circle of convergence extends up to the nearest singular point. As we shall see, no such simple relation holds in the general case; a Dirichlet's series convergent in a portion of the plane only may represent a function regular all over the plane, or in a wider region of

* If $e^{\delta\lambda_n} > 2$ for $n \geqq n_0$, as we can obviously suppose.

it. The result is (to put it roughly) that many of the peculiar difficulties which attend the study of power series *on* the circle of convergence are extended, in the case of Dirichlet's series, to wide regions of the plane or even to the whole of it.

There is however one important case in which the line of convergence necessarily contains at least one singularity.

THEOREM 10. *If all the coefficients of the series are positive or zero, then the real point of the line of convergence is a singular point of the function represented by the series**.

We may suppose that $\sigma_0 = \bar{\sigma} = 0$. Then, if $s = 0$ is a regular point, the Taylor's series for $f(s)$, at the point $s = 1$, has a radius of convergence greater than 1. Hence we can find a negative value of s for which

$$f(s) = \sum_{\nu=0}^{\infty} \frac{(s-1)^\nu}{\nu!} f^{(\nu)}(1) = \sum_{\nu=0}^{\infty} \frac{(1-s)^\nu}{\nu!} \sum_{n=1}^{\infty} a_n \lambda_n^\nu e^{-\lambda_n}.$$

But every term in this repeated series is positive. Hence† the order of summation may be inverted, and we obtain

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n} \sum_{\nu=0}^{\infty} \frac{(1-s)^\nu \lambda_n^\nu}{\nu!} = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}.$$

Thus the series is convergent for some negative values of s, which contradicts our hypotheses.

In the general case all conceivable hypotheses may actually be realised. Thus the series

$$1^{-s} - 2^{-s} + 3^{-s} - \ldots,$$

which converges for $\sigma > 0$, represents the function

$$(1 - 2^{1-s})\,\zeta(s),\ddagger$$

which is regular all over the plane. The series

$$\sum 2^{-2^n s}$$

has the imaginary axis as a line of essential singularities§.

* This theorem was proved first for power series by Vivanti, **1**, and Pringsheim, **1**. It was extended to the general case by Landau, **1**, and *H.*, p. 880. Further interesting generalisations have been made by Fekete, **1**, **2**.

† Bromwich, *Infinite series*, p. 78.

‡ For the theory of the famous ζ-function of Riemann, we must refer to Landau's *Handbuch* and the *Cambridge Tract* by Messrs Bohr and Littlewood which, we hope, is to follow this.

§ Landau, **2**. General classes of such series have been defined by Knopp **4**. Schnee, **1**, **3**, and Knopp, **1**, **3**, **5**, have also given a number of interesting theorems relating to the behaviour of $f(s)$ as s approaches a singular point on the line of convergence, the coefficients of the series being supposed to obey certain asymptotic laws. These theorems constitute a generalisation of the work of Appell, Cesàro, Lasker, Pringsheim and others on power series.

9. **Representation of a Dirichlet's series as a definite integral.** We may mention here the following theorem, which is interesting in itself and useful in the study of particular series. We shall not use it in this tract, and therefore do not include a proof.

THEOREM 11. *Let* $\mu_n = \log \lambda_n$. *Then*

$$\Sigma a_n e^{-\mu_n s} = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} (\Sigma a_n e^{-\lambda_n x})\, dx$$

if $\sigma > 0$ *and the series on the left-hand side is convergent**.

We have, for example,

$$\zeta(s) = \Sigma n^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx, \quad (1 - 2^{1-s})\,\zeta(s) = \Sigma(-1)^{n-1} n^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x + 1}\, dx.$$

Here $\sigma > 1$ in the first formula, and $\sigma > 0$ in the second; and $\zeta(s)$ is the Riemann ζ-function.

III

THE FORMULA FOR THE SUM OF THE COEFFICIENTS OF A DIRICHLET'S SERIES: THE ORDER OF THE FUNCTION REPRESENTED BY THE SERIES

1. We shall now prove a theorem which is of fundamental importance for the later developments of the theory.

THEOREM 12†. *Suppose* $\lambda_1 \geqq 0$ *and the series convergent or finitely oscillating for* $s = \beta$. *Then*

$$\sum_1^n a_\nu e^{-\lambda_\nu s} = o\,|t|$$

uniformly for $\sigma \geqq \beta + \epsilon > \beta$ *and all values of* n; *that is to say, given any positive numbers* δ, ϵ, *we can find* t_0 *so that*

$$\left| \frac{1}{t} \sum_1^n a_\nu e^{-\lambda_\nu s} \right| < \delta$$

for $\sigma \geqq \beta + \epsilon$, $|t| \geqq t_0$, *and all values of* n. *In particular we have, for* $n = \infty$,

$$f(s) = o\,|t|$$

uniformly for $\sigma \geqq \beta + \epsilon$.

* See Cahen, 1; Perron, 1; Hardy, 5; the last two authors give rigorous proofs.

† Landau, *H.*, p. 821.

We may take $\beta = 0$ without loss of generality. Then

$$|a_\nu| < K, \quad |A(\mu, \nu)| < K$$

for all values of μ and ν. Also, if $1 < N < n$, we have

$$\sum_1^n a_\nu e^{-\lambda_\nu s} = \sum_1^{N-1} a_\nu e^{-\lambda_\nu s} + \sum_N^{n-1} A(N, \nu)\, \Delta e^{-\lambda_\nu s} + A(N, n)\, e^{-\lambda_n s}$$

$$= S_1 + S_2 + S_3,$$

say; and since $|e^{-\lambda_n s}| < 1$ if $\sigma \geqq \epsilon$, we have

$$|S_1| < KN, \quad |S_3| < K, \quad S_1 + S_3 = O(N).$$

We have moreover, by Lemma 2 of II, § 1,

$$|S_2| < K\frac{|s|}{\sigma}\sum_N^{n-1} \Delta e^{-\lambda_\nu \sigma} < K\sqrt{\left(1 + \frac{t^2}{\epsilon^2}\right)}\, e^{-\lambda_N \epsilon},$$

$$\sum_1^n a_\nu e^{-\lambda_\nu s} = O(N) + O(te^{-\lambda_N \epsilon})$$

if $1 < N < n$. On the other hand it is evident that

$$\sum_1^n a_\nu e^{-\lambda_\nu s} = O(N)$$

if $N \geqq n$. If now we suppose that N is a function of $|t|$ which tends to infinity more slowly than $|t|$, we see that in any case

$$\sum_1^n a_\nu e^{-\lambda_\nu s} = o\,|t|.$$

2. We now apply Theorem 12 to prove an important theorem first rigorously and generally established by Perron*.

THEOREM 13. *If the series is convergent for* $s = \beta + i\gamma$, *and*

$$c > 0, \quad c > \beta, \quad \lambda_n < \omega < \lambda_{n+1},$$

then

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\, e^{\omega s}\, \frac{ds}{s} = \sum_1^n a_\nu,$$

the path of integration being the line $\sigma = c$. *At a point of discontinuity* $\omega = \lambda_n$, *the integral has a value half-way between its limits on either side, but in this case the integral must be regarded as being defined by its principal value*†.

* Perron, **1**. See also Cahen, **1**; Hadamard, **1** (where a rigorous proof is given for series which possess a half-plane of absolute convergence); Landau, *H.*, pp. 820 *et seq.*

† The *principal value* is the limit, if it exists, of

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(s)\, e^{\omega s}\, \frac{ds}{s},$$

which may exist when the integral, as ordinarily defined, does not.

This theorem depends upon the following lemma.

LEMMA 3. *If x is real, we have*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{xs}\frac{ds}{s} \begin{cases} =1 & (x>0), \\ =\tfrac{1}{2} & (x=0), \\ =0 & (x<0), \end{cases}$$

it being understood that in the second case the principal value of the integral is taken.

We may leave the verification of this result as an exercise to the reader*.

Let $\lambda_n < \omega < \lambda_{n+1}$ and

$$g(s) = e^{\omega s}\left\{f(s) - \sum_1^n a_\nu e^{-\lambda_\nu s}\right\} = \sum_{n+1}^{\infty} a_\nu e^{-(\lambda_\nu - \omega)s}$$
$$= \sum_1^{\infty} b_\nu e^{-\mu_\nu s},$$

where $b_\nu = a_{n+\nu}$, $\mu_\nu = \lambda_{n+\nu} - \omega$, so that $\mu_1 > 0$. It is clear from the lemma that what we have to show is that

$$\int_{c-i\infty}^{c+i\infty} g(s)\frac{ds}{s} = 0 \quad \text{......................(1).}$$

Applying Cauchy's theorem to the rectangle whose vertices are

$$c - iT_1,\ c + iT_2,\ \gamma + iT_2,\ \gamma - iT_1,$$

where T_1 and T_2 are positive, and $\gamma > c$, we obtain

$$\int_{c-iT_1}^{c+iT_2} g(s)\frac{ds}{s} = \int_{c-iT_1}^{\gamma-iT_1} g(s)\frac{ds}{s} - \int_{c+iT_2}^{\gamma+iT_2} g(s)\frac{ds}{s} + \int_{\gamma-iT_1}^{\gamma+iT_2} g(s)\frac{ds}{s}.$$

Keeping T_1 and T_2 fixed, we make γ tend to infinity. By Theorem 2, the upper limit of $|g(s)|$ in the last integral remains, throughout this process, less than a number independent of γ. Hence the last integral tends to zero, and

$$\int_{c-iT_1}^{c+iT_2} g(s)\frac{ds}{s} = \int_{c-iT_1}^{\infty-iT_1} g(s)\frac{ds}{s} - \int_{c+iT_2}^{\infty+iT_2} g(s)\frac{ds}{s} \text{.........(2),}$$

if the two integrals on the right-hand side are convergent. Now, if we write

$$g(s) = e^{-\mu_1 s} h(s),$$

* The easiest method of verification is by means of Cauchy's Theorem. Full details will be found in Landau, *H.*, pp. 342 *et seq.*

we can, by Theorem 12, choose T so that $|h(s)| < \epsilon T_2$ for $s = \sigma + iT_2$, $\sigma \geqq c$, $T_2 > T$. Hence the second integral on the right-hand side of (2) is convergent, and

$$\left| \int_{c+iT_2}^{\infty+iT_2} g(s) \frac{ds}{s} \right| < \frac{\epsilon T_2}{\sqrt{(c^2 + T_2^2)}} \int_c^\infty e^{-\mu_1 \xi} d\xi < \frac{\epsilon}{\mu_1}.$$

Thus the integral in question tends to zero as $T_2 \to \infty$. Similarly for the integral involving T_1. Hence (1) is established and the theorem is proved, except when ω is equal to one of the λ's. The reader will have no difficulty in supplying the modifications necessary in this case.

3. The order of $f(s)$ for $s = \beta$ and for $s \geqq \beta$. Theorem 12 suggests the introduction of an idea which will be prominent in the rest of this tract.

Suppose that $f(s)$ is a function of s regular for $\sigma > \gamma$. If $\beta > \gamma$, and ξ is any real number, it may or may not be true that

$$f(\sigma + ti) = O(|t|^\xi) \quad \text{.......................(1)},$$

when $\sigma = \beta$ and $|t| \to \infty$. If this equation is true for a particular value of ξ, it is true for any greater value. It follows, by a classical argument, that there are three possibilities. The equation (1) may be true for all values of ξ, or for some but not all, or for none. In the second case there is a number μ such that (1) is true for $\xi > \mu$ and untrue for $\xi < \mu$. In the first case we may agree to write conventionally $\mu = -\infty$, and in the third case $\mu = \infty$. We thus obtain a function $\mu(\sigma)$ defined for $\sigma > \gamma$; and we call $\mu(\beta)$ *the order of $f(s)$ for $\sigma = \beta$*. When it is not true that $\mu(\beta) = \infty$, we say that $f(s)$ is *of finite order for $\sigma = \beta$*.

Again, the equation (1) may or may not hold uniformly for $\sigma \geqq \beta$. If we consider it from this point of view, and apply exactly the same arguments as before, we are led to define a function $\nu(\beta)$ which we call *the order of $f(s)$ for $\sigma \geqq \beta$*. Evidently $\nu \geqq \mu$. When it is not true that $\nu(\beta) = \infty$, we say that $f(s)$ is *of finite order for $\sigma \geqq \beta$*. And if $f(s)$ is of finite order for $\sigma \geqq \beta + \epsilon$, for every positive ϵ, but not necessarily for $\sigma \geqq \beta$, we shall say that it is *of finite order for $\sigma > \beta$*. Finally, the equation (1), without holding uniformly for $\sigma \geqq \beta_1$, may hold uniformly for $\beta_1 \leqq \sigma \leqq \beta_2$. We are thus led to define the order of $f(s)$ for $\beta_1 \leqq \sigma \leqq \beta_2$. The reader will find no difficulty in framing a formal definition, or in giving precise interpretations of the phrases '*$f(s)$ is of finite order for $\beta_1 \leqq \sigma \leqq \beta_2$*', '*$f(s)$ is of finite order for $\beta_1 < \sigma < \beta_2$*'.

4. Lindelöf's Theorem. In order to establish the fundamental properties of the function $\mu(\sigma)$ associated with a function $f(s)$, defined initially by a Dirichlet's series, we shall require the following theorem, which is due to Lindelöf, and is one of a class of general theorems the first of which were discovered by Phragmén *.

THEOREM 14. *If* (i) $f(s)$ *is regular and of finite order for* $\beta_1 \leqq \sigma \leqq \beta_2$, (ii) $f(s) = O(|t|^{k_1})$ *for* $\sigma = \beta_1$, (iii) $f(s) = O(|t|^{k_2})$ *for* $\sigma = \beta_2$, *then*

$$f(s) = O(|t|^{k(\sigma)}),$$

uniformly for $\beta_1 \leqq \sigma \leqq \beta_2$, $k(x)$ *being the linear function of* x *which assumes the values* k_1, k_2 *for* $x = \beta_1$, β_2.

The special case in which $k_1 = k_2 = 0$ is of particular interest; we have then the result that *if* $f(s)$ *is of finite order for* $\beta_1 \leqq \sigma \leqq \beta_2$, *and bounded on the lines* $\sigma = \beta_1$ *and* $\sigma = \beta_2$, *then it is bounded in the whole strip between them.*

In proving this theorem we may evidently confine our attention to positive values of t.

First, suppose k_1 as well as k_2 to be zero, so that $k(x)$ is identically zero and $f(s) = O(1)$ for $\sigma = \beta_1$ and $\sigma = \beta_2$. Let M be the upper bound of the values of $|f|$ on these two lines and the segment (β_1, β_2) of the real axis. Also let

$$g(s) = e^{\epsilon s i} f(s) \qquad (\epsilon > 0).$$

Then
$$|g(s)| = e^{-\epsilon t} |f(s)| \leqq |f(s)|,$$

so that $g(s) = O(1)$ for $\sigma = \beta_1$ and $\sigma = \beta_2$. Also, as f is of finite order, $g \to 0$ as $t \to \infty$, uniformly for $\beta_1 \leqq \sigma \leqq \beta_2$. Hence, when ϵ is given, we can determine t_0 so that $|g| < M$ for $\beta_1 \leqq \sigma \leqq \beta_2$, $t > t_0$. It follows that any point whose abscissa lies between β_1 and β_2 can be surrounded by a contour at each point of which $|g| < M$, the contour being a rectangle formed by the lines $\sigma = \beta_1$, $\sigma = \beta_2$, the real axis, and a parallel to it at a sufficiently great distance from the origin. Hence, by a well-known theorem, $|g| < M$ at the point itself, and so

$$|f(s)| < Me^{\epsilon t}.$$

This is true for all positive values of ϵ, and therefore $|f| \leqq M$. Thus the theorem is proved. It should be observed that, if we had

* Lindelöf, **1**. See also Phragmén, **1**; Phragmén and Lindelöf, **1**; Landau, *H.*, pp. 849 *et seq.*

used the factor $e^{\epsilon s^2}$ instead of $e^{\epsilon si}$, we could have proved a little more, viz. that if $|f(s)|$ is less than M for $\sigma = \beta_1$ and $\sigma = \beta_2$, it is less than M for $\beta_1 \leqq \sigma \leqq \beta_2$. We leave the formal proof of this as an exercise for the reader.

Next, suppose $k(x)$ not identically zero, and consider the function*

$$h(s) = (-si)^{k(s)} = e^{k(s)\log(-si)},$$

where the logarithm has its principal value. This function is regular in the region $\beta_1 \leqq \sigma \leqq \beta_2$, $t \geqq 1$,† and, within this region, may be expressed in the form

$$e^{\{k(\sigma)+cti\}\{\log t + O(1/t)\}},$$

where c is a real constant. Thus

$$|h(s)| = t^{k(\sigma)} e^{O(1)},$$

so that the ratio of $|h(s)|$ to $t^{k(\sigma)}$ remains throughout the region between fixed positive limits. Hence the function

$$F(s) = f(s)/h(s)$$

satisfies the conditions which we supposed before to be satisfied by $f(s)$. Thus

$$F(s) = O(1), \quad f(s) = O\{t^{k(\sigma)}\},$$

uniformly throughout the region; and the theorem is completely proved.

5. Properties of the function $\mu(\sigma)$ associated with a Dirichlet's series which has a domain of absolute convergence. We shall now apply Lindelöf's Theorem to establish the fundamental properties of the function $\mu(\sigma)$, when $f(s)$ is defined by a Dirichlet's series. In order to obtain simple and definite results, we shall limit ourselves to the case in which there is a domain of absolute convergence.

THEOREM 15. *Suppose that the series $\Sigma a_n e^{-\lambda_n s}$ is absolutely convergent for $\sigma > \bar{\sigma}$, and that the function $f(s)$, defined by the series when $\sigma > \bar{\sigma}$, is regular and of finite order for $\sigma > \gamma$, where $\gamma < \bar{\sigma}$. Then the function $\mu(\sigma)$, defined for $\sigma > \gamma$, has the following properties. Either it is always zero; or it is zero for $\sigma \geqq \gamma_0$, where $\gamma < \gamma_0 \leqq \bar{\sigma}$, while*

* This auxiliary function (introduced by Landau, **9**) is a little simpler than that used by Lindelöf.

† We suppose $t \geqq 1$ instead of, as before, $t \geqq 0$, to avoid the singularity of $h(s)$ for $s = 0$ even when $\beta_1 \leqq 0 \leqq \beta_2$. In proving the theorem it is evidently only necessary to consider values of t greater than some fixed value.

for $\gamma < \sigma < \gamma_0$ *it is a positive, decreasing, convex*, and continuous function of* σ. *Further,* $\nu(\sigma)$ *is identical with* $\mu(\sigma)$.†

Suppose that $\mu = \mu_1$ for $\sigma = \beta_1 > \gamma$, and $\mu = \mu_2$ for $\sigma = \beta_2 > \beta_1$. Then

$$f(\beta_1 + ti) = O(|t|^{\mu_1 + \epsilon_1}), \quad f(\beta_2 + ti) = O(|t|^{\mu_2 + \epsilon_2}),$$

where ϵ_1, ϵ_2 are any positive numbers. Applying Lindelöf's Theorem we obtain at once

$$\mu \leqq \frac{\beta_2 - \sigma}{\beta_2 - \beta_1}(\mu_1 + \epsilon_1) + \frac{\sigma - \beta_1}{\beta_2 - \beta_1}(\mu_2 + \epsilon_2),$$

or, as ϵ_1 and ϵ_2 are arbitrarily small,

$$\mu \leqq \frac{(\beta_2 - \sigma)\mu_1 + (\sigma - \beta_1)\mu_2}{\beta_2 - \beta_1} \quad \text{.....................(1)},$$

for $\beta_1 \leqq \sigma \leqq \beta_2$. This relation expresses the fundamental property of the function μ.

A similar argument shows that, if $\mu = -\infty$ for any σ, the same must be true for every σ. We shall see in a moment that this possibility may, in the present case, be ignored.

It is clear that $\mu \leqq 0$ for $\sigma > \bar{\sigma}$; for if $\beta > \bar{\sigma}$ then

$$|f(s)| < \Sigma |a_n| e^{-\lambda_n \beta}$$

for $\sigma \geqq \beta$. But it is easy to see also that $\mu \geqq 0$ for sufficiently large values of σ. For, if a_m is the first coefficient in the series which does not vanish, we may write $f(s)$ in the form

$$a_m e^{-\lambda_m s} + e^{-\lambda_m s} \sum_{n>m} a_n e^{-(\lambda_n - \lambda_m)s}.$$

The series here written is absolutely and uniformly convergent for $\sigma > \bar{\sigma}$, and so tends uniformly to zero as $\sigma \to \infty$. Hence we can so choose ω that

$$f(s) = a_m e^{-\lambda_m s}(1 + \rho),$$

* We say that $f(x)$ is *convex* if it satisfies the inequality

$$f\{\theta x + (1-\theta)y\} \leqq \theta f(x) + (1-\theta) f(y)$$

for $0 \leqq \theta \leqq 1$. The theory of such functions has been investigated systematically by Jensen, **3**. If we put $\theta = \frac{1}{2}$ we obtain the inequality

$$2f\{\tfrac{1}{2}(x+y)\} \leqq f(x) + f(y);$$

and Jensen has shown that, if $f(x)$ is continuous, the more general inequality can be deduced from this. A continuous function is certainly convex if

$$\lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

exists for all values of x and is never negative. See Harnack, **1**, and Hölder, **1**.

† The results comprised in Theorem 15 are in the main due to Bohr, **5**.

where $|\rho| < \frac{1}{2}$ for $\sigma > \omega$. Thus $|f|$ has a positive lower bound when σ has a fixed value greater than ω and $|t| \to \infty$; and so $\mu \geqq 0$, and therefore $\mu = 0$, for $\sigma > \omega$.

We can now show that μ can never be negative. For if μ were ever negative we could suppose, in (1), that $\mu_1 < 0$, while β_2 and σ are both greater than ω, so that $\mu_2 = 0$, $\mu = 0$. This obviously involves a contradiction. We thus see that $\mu = 0$ for $\sigma > \bar{\sigma}$.

Again, if in (1) we suppose $\mu_1 > 0$, and $\beta_2 > \bar{\sigma}$, so that $\mu_2 = 0$, we see that $\mu < \mu_1$ if $\sigma > \beta_1$. Thus μ, in so far as it is not zero, is a decreasing function of σ, in the stricter sense which forbids equality of values.

The only property of $\mu(\sigma)$ which remains to be established is its continuity. If β_1 is a particular value of σ, the numbers

$$\mu(\beta_1 - 0) = \mu_1', \quad \mu(\beta_1) = \mu_1, \quad \mu(\beta_1 + 0) = \mu_1''$$

all exist (since μ is monotonic) and $\mu_1' \geqq \mu_1 \geqq \mu_1''$. But since μ is convex, we have the inequalities

$$\mu(\beta_1 - \delta) \leqq \mu(\beta_1 - 2\delta) - \mu(\beta_1 - \delta) + \mu(\beta_1),$$
$$\mu(\beta_1) - \mu(\beta_1 + \delta) \leqq \mu(\beta_1 - \delta) - \mu(\beta_1),$$

where δ is positive. Making $\delta \to 0$, we obtain from the first $\mu_1' \leqq \mu_1$, and then from the second $\mu_1 \leqq \mu_1''$. It follows that $\mu_1' = \mu_1 = \mu_1''$, so that μ is continuous.

Finally, in order to establish the equivalence of the functions μ and ν, it is only necessary to apply Theorem 14 to a strip whose right-hand edge lies in the domain of absolute convergence, and to take account of the uniformity asserted by the theorem. This we may leave to the reader.

We may remark that it follows from Theorem 12 that $\mu \leqq 1$ for $\sigma > \sigma_0$, where σ_0 is the abscissa of convergence.

6. The actual determination of the function $\mu(\sigma)$ associated with a given Dirichlet's series is in general a problem of extreme difficulty Consider for example the series

$$1^{-s} - 2^{-s} + 3^{-s} - \ldots = (1 - 2^{1-s})\,\zeta(s).$$

The series is convergent for $\sigma > 0$, and absolutely convergent for $\sigma > 1$; the function is regular all over the plane. Obviously $\mu(\sigma) = 0$ for $\sigma \geqq 1$, while it may easily be shown, by means of Riemann's functional equation for the ζ-function, that $\mu(\sigma) = \frac{1}{2} - \sigma$ for $\sigma \leqq 0$. All that is known about $\mu(\sigma)$ for $0 \leqq \sigma \leqq 1$ is that its graph does not rise above the line joining the points $(0, \frac{1}{2})$ and $(1, 0)$*. It has however been proved by Littlewood† that, if it be true that all the complex roots of $\zeta(s)$ have the real part $\frac{1}{2}$, then

$$\mu(\sigma) = \tfrac{1}{2} - \sigma \quad (\sigma < \tfrac{1}{2}), \quad \mu(\sigma) = 0 \quad (\sigma \geqq \tfrac{1}{2}).$$

* Lindelöf, **1**; Landau, *H.*, p. 868.

† Littlewood, **2**.

7. Let us consider now a Dirichlet's series with distinct lines of convergence and absolute convergence. To fix our ideas let us suppose $\sigma_0 = 0$, $\bar{\sigma} = 1$, and the function regular and of finite order for some negative values of σ. Then $\mu = 0$ for $\sigma = 1$, and $\mu \leqq 1$ for $\sigma = 0$. It follows at once, from the convexity of μ, that $\mu \leqq 1 - \sigma$ for $0 \leqq \sigma \leqq 1$. It will be proved later on* that $\mu > 0$ for $\sigma < 0$. Thus the final range of invariability of μ, which cannot begin later than $\sigma = 1$, cannot begin earlier than $\sigma = 0$. Bohr† has constructed examples which show that the two extreme cases thus indicated can actually occur. He has shown that it is possible to find two ordinary Dirichlet's series for each of which $\sigma_0 = 0$, $\bar{\sigma} = 1$, while for one series $\mu = 0$ for $\sigma > 0$ and for the other $\mu = 1 - \sigma$ for $0 < \sigma < 1$. He has also shown‡ that it is possible for two ordinary Dirichlet's series to have the same μ-function but different regions of convergence, and so that it is futile to attempt to define the region of convergence of a Dirichlet's series in terms merely of the associated μ-function.

So far we have assumed the existence of a domain of absolute convergence. Some of our arguments remain valid in the general case, but it is no longer possible to obtain such simple and satisfactory results. We shall content ourselves, therefore, with mentioning one further interesting result of Bohr. Suppose that the indices λ_n are *linearly independent*§, that is to say that there are no relations of the type

$$k_1\lambda_1 + k_2\lambda_2 + \ldots + k_n\lambda_n = 0,$$

where the k's are integers, not all zero, holding between them. Then Bohr‖ has shown that, if $f(s)$ is regular and *bounded* for $\sigma \geqq \beta$, the series is absolutely convergent for $\sigma \geqq \beta$. Thus if there is no region of absolute convergence, the function cannot be bounded in any half-plane. He has also shown by an example that this conclusion is no longer necessarily correct when the restriction of the linear independence of the λ's is removed.

IV

THE SUMMATION OF SERIES BY TYPICAL MEANS

1. So far we have considered only *convergent* Dirichlet's series. We have seen that such a series defines an analytical function which may or may not exist outside of the domain of convergence of the series.

* See VII, § 10, Theorem 50. † Bohr, **5**, pp. 30 *et seq.*

‡ *l.c. supra* p. 36.

§ This is in a sense the *general* case. The condition is satisfied, for example, when $\lambda_n = \log p_n$, where p_n is the n-th prime, but not when $\lambda_n = n$ or $\lambda_n = \log n$. The *result* of course still holds when $\lambda_n = n$.

‖ Bohr, **7**.

In the modern developments of the theory of power series a great part has been played by a variety of methods of summation of oscillating series, which we associate with the names of Frobenius, Hölder, Cesàro, Borel, Lindelöf, Mittag-Leffler, and Le Roy*. Of these definitions the simplest and the most natural is that which defines the sum of an oscillating series as the limit of the arithmetic mean of its first n partial sums. This definition was generalised in two different ways by Hölder and by Cesàro, who thus arrived at two systems of definitions the complete equivalence of which has been established only recently by Knopp, Schnee, Ford, and Schur†.

The range of application of Cesàro's methods is limited in a way which forbids their application to the problem of the analytical continuation of the function represented by a Taylor's series. A power series, outside its circle of convergence, diverges too crudely for the application of such methods: more powerful, though less delicate, methods, such as Borel's ‡, are required. But Cesàro's methods have proved of the highest value in the study of power series *on* the circle of convergence and the closely connected problems of the theory of Fourier's series§. And it is natural to suppose that in the theory of Dirichlet's series, where we are dealing with series whose convergence or divergence is of a much more delicate character than is, in general, that of a power series, they will find a wider field of application.

The first such applications were made independently by Bohr and Riesz‖, who showed that the arithmetic means formed in Cesàro's manner from an ordinary Dirichlet's series may have domains of convergence more extensive than that of the series itself¶. But it appeared from the investigations of Riesz that these arithmetic means

* For a general account of some of these methods and the relations between them, see Borel, *Leçons sur les séries divergentes*, Ch. 3; Bromwich, *Infinite series*, Ch. 11; Hardy and Chapman, **1**; Hardy and Littlewood, **1**, **2**.

† Knopp, **1**; Schnee, **2**; Ford, **1**; Schur, **1**. See also Bromwich, **1**; Faber, **1**; Landau, **13**; Knopp, **7**.

‡ Cf. Hardy and Littlewood, **1**.

§ We need only mention Fejér's well-known theorem (Fejér, **1**) and its generalisations by Lebesgue, **1**; Riesz, **3**; Chapman, **1**, **2**; Young, **1**, **2**; Hardy and Littlewood, **3**; and Hardy, **9**. We should say here that when referring to Marçel Riesz we write simply 'Riesz'.

‖ Bohr, **1**, **2**, **5**, **6**; Riesz, **1**, **2**, **3**, **4**.

¶ Thus, as was shown in a very simple manner by Bohr, **5**, the series

$$\Sigma(-1)^{n-1}n^{-s}$$

is summable by Cesàro's k-th mean if $\sigma > -k$. In so far as real values of s are concerned, this had already been proved by Cesàro, in a somewhat less elementary way (see Bromwich, *Infinite series*, p. 317). The general result follows from this and Theorem 29 below.

are not so well adapted to the study of the series as certain other means formed in a somewhat different manner. These 'logarithmic means'*, as well as the arithmetic means, have generalisations especially adapted to the study of the general series $\Sigma a_n e^{-\lambda_n s}$. We shall begin by giving the formal definitions of these means†; we shall then indicate shortly how they form a natural generalisation of Cesàro's.

2. Definitions. We suppose $\lambda_1 \geqq 0$, and we write

$$e^{\lambda_n} = l_n, \quad a_n e^{-\lambda_n s} = a_n l_n^{-s} = c_n,$$

$$C_\lambda(\tau) = \sum_{\lambda_n < \tau} c_n, \quad C_l(t) = \sum_{l_n < t} c_n.$$

Thus
$$C_\lambda(\tau) = c_1 + c_2 + \ldots + c_n \qquad (\lambda_n < \tau \leqq \lambda_{n+1}).$$
If $\lambda_n = n$, $C_\lambda(\tau)$ is identical with the function $C(\tau)$ defined in I, § 2, except when τ is an integer n, when the two functions differ by c_n.‡

Further, we shall write

$$C_\lambda^\kappa(\omega) = \sum_{\lambda_n < \omega} (\omega - \lambda_n)^\kappa c_n = \kappa \int_0^\omega C_\lambda(\tau)(\omega - \tau)^{\kappa - 1} d\tau,$$

where κ is any positive number, integral or not§. We leave it to the reader to verify the equivalence of the two expressions of $C_\lambda^\kappa(\omega)$. In a precisely similar way we define $C_l^\kappa(w)$: thus

$$C_l^\kappa(w) = \sum_{l_n < w} (w - l_n)^\kappa c_n = \kappa \int_1^w C_l(t)(w - t)^{\kappa - 1} dt. \|$$

We shall call the functions

$$C_\lambda^\kappa(\omega)/\omega^\kappa, \quad C_l^\kappa(w)/w^\kappa,$$

introduced by Riesz, the *typical means* (*moyennes typiques*¶) *of order* κ,

* Riesz, **2**.

† Riesz, **3**. The definitions of this note, which are those which we adopt as final, differ from those of the earlier note.

‡ If $\lambda(x)$ is an increasing function of x, which assumes the value λ_n for $x = n$, then $C_\lambda(\lambda)$, regarded as a function of x, is, except for integral values of x, the same function as $C(x)$.

§ The theory of non-integral orders of summation by means of Cesàro's type has been developed by Knopp, **1**, **2**, and by Chapman, **1**. These writers consider also *negative* orders of summation (greater than -1): but we shall not be concerned with such negative orders here. It should be observed that $C_\lambda^\kappa(\omega) = 0$ if $\omega \leqslant \lambda_1$.

‖ We have $C_l(t) = 0$ if $t \leqq l_1$; and $l_1 = e^{\lambda_1} \geqslant 1$. Thus we may take 1 as the lower limit. And $C_l^\kappa(w) = 0$ if $w \leqslant l_1$.

¶ Riesz, **3**. The type is the type of the associated Dirichlet's series (I, § 1).

of the first and the second kind, associated with the series $\Sigma a_n e^{-\lambda_n s}$. It should be observed that, so long as we are thinking merely of the problem of summing a numerical series Σc_n, the word 'typical' is devoid of significance; it only acquires a significance when we are summing a Dirichlet's series of a special type by means specially defined with reference to that type. We shall frequently omit the suffixes λ, l, when no ambiguity can arise from so doing. The reader must however bear in mind the distinction already referred to between the function $C(\tau)$ thus defined and the function $C(\tau)$ of Section I.

If $$\omega^{-\kappa} C_\lambda^\kappa(\omega) \to C$$

as $\omega \to \infty$, *we shall say that the series* Σc_n *is summable* (λ, κ) *to sum* C*. If the typical mean oscillates finitely as $\omega \to \infty$, we shall say that the series is *finite* (λ, κ). Similarly we define summability and finitude (l, κ).

We add a few remarks† to show the genesis of these definitions. If $\lambda_n = n$, the typical mean of the first kind is

$$\omega^{-\kappa} \sum_{n<\omega} (\omega - n)^\kappa c_n = \kappa\, \omega^{-\kappa} \int_0^\omega C(\tau)(\omega-\tau)^{\kappa-1}\, d\tau.$$

If in particular $\kappa = 1$, and ω is an integer, we obtain

$$\frac{1}{\omega} \sum_1^{\omega-1} (\omega - n)\, c_n = \frac{C_1 + C_2 + \ldots + C_{\omega-1}}{\omega}$$

(where $C_n = c_1 + c_2 + \ldots + c_n$), which is practically Cesàro's first mean. If κ is an integer greater than unity, we have

$$\kappa\omega^{-\kappa} \int_0^\omega C(\tau)(\omega-\tau)^{\kappa-1}\, d\tau = \kappa\,!\, \omega^{-\kappa} \left(\int_0 d\tau\right)^\kappa C(\tau).‡$$

Now Cesàro's κ-th mean is

$$\kappa\,!\, n^{-\kappa} C_n^\kappa,$$

where C_n^κ is the κ-th repeated sum formed from the numbers C_n, and it is plain that, as soon as we replace C_n by a function $C(\tau)$ of a continuous variable τ, we are naturally led to the definitions adopted here. And it is then also natural to abandon the restriction that κ is an integer. The integrals to which we are thus led are of course of the type employed by Liouville and Riemann in their theories of non-integral orders of differentiation and integration §.

* This is substantially the notation introduced by Hardy, **4**. Hardy writes '*summable* (R, λ, κ)'—*i.e.* summable by Riesz's means of type λ and order κ.

† Compare Hardy, **6**.

‡ See, *e.g.*, Jordan, *Cours d'analyse*, Vol. 3, p. 59.

§ Liouville, **1**, **2**; Riemann, **1**. See also Borel, *Leçons sur les séries à termes positifs*, pp. 74 *et seq.*

It has in fact been shown by Riesz* that these definitions are completely equivalent to Cesàro's, and to the generalisations of Cesàro's considered by Knopp and Chapman†. It is this which justifies our calling the typical means of the first kind, when $\lambda_n = n$, *arithmetic* means. In this case the means of the second kind are not of any interest‡.

So much for the case when $\lambda_n = n$. If in the general definition we put $\kappa = 1$, $\omega = \lambda_n$, we obtain

$$(\mu_1 C_1 + \mu_2 C_2 + \ldots + \mu_{n-1} C_{n-1})/\lambda_n,$$

where $\mu_\nu = \lambda_{\nu+1} - \lambda_\nu$. This is the natural generalisation of Cesàro's first mean which suggests itself when we try to attach varying weights to the successive partial sums C_ν.

When $\lambda_n = \log n$, the series $\Sigma a_n e^{-\lambda_n s}$ is an ordinary Dirichlet's series. The means of the first kind are then what Riesz has called *logarithmic means* §, and it is the means of the second kind that are arithmetic means. From the theoretical standpoint, the former are in general better adapted to the study of ordinary Dirichlet's series‖. On the other hand, arithmetic means are simpler in form and often easier to work with. Hence it is convenient and indeed necessary to take account of both kinds of means; and the same is true, of course, in the general theory.

3. Summable Integrals. It is easy to frame corresponding definitions for integrals. We suppose that $\lambda(x)$ is a positive and continuous function of x, which tends steadily to infinity with x, and that $\lambda(0) = 0$, and write

$$C_\lambda(x) = \int_{\lambda(u) < x} c(u)\, du = \int_0^{\bar{\lambda}(x)} c(u)\, du,$$

where $\bar{\lambda}$ is the function inverse to λ. Further we write

$$C_\lambda^\kappa(\omega) = \kappa \int_0^\omega C_\lambda(x)(\omega - x)^{\kappa - 1}\, dx.$$

Then *if*

$$\omega^{-\kappa} C_\lambda^\kappa(\omega) \to C,$$

as $\omega \to \infty$, *we shall say that the integral*

$$\int_0^\infty c(x)\, dx$$

is summable (λ, κ) *to sum* C.

This definition may be applied to the theory of integrals of the type

$$\int_0^\infty a(x)\, e^{-s\lambda(x)}\, dx$$

* Riesz, **4**.

† Knopp, **1**, **2**; Chapman, **1**.

‡ See § 4 (3).

§ Riesz, **2**.

‖ It may also be observed that the form of the arithmetic means which we have adopted is better adapted for this purpose than that of Cesàro. Thus, for example, Schnee (**7**, pp. 393 *et seq.*), working with Cesàro's means, was able to avoid an unnecessary restriction only by using the result of Riesz cited above as to the equivalence of Cesàro's means and our 'arithmetic' means. See p. 56, footnote (*).

in the same way that the definitions of § 2 may be applied to the theory of Dirichlet's series*.

Most of the theorems which we shall prove for series have their analogues for integrals. When this is so, the proofs are in general easier for integrals than for series, and we have not thought it worth while to give the details of any of them. We have not even stated the theorems themselves explicitly, except one theorem (V, § 3) of which we shall make several applications.

4. In this paragraph we shall state, without detailed proofs, a number of special results which may be regarded as exercises for the reader. Some of them are in reality special cases of general theorems which we shall give later on. They are inserted here, partly because they are interesting in themselves, and partly in order to familiarise the reader with our notation and to give him a general idea of the range of our definitions.

(1) If Σa_n is convergent and has the sum A, then

$$\Sigma a_n e^{-\lambda_n s} = A + \int_0^\infty s e^{-su} \{A_\lambda(u) - A\}\, du$$

for all values of s for which the series is convergent. If in addition $\sigma > 0$, then

$$\Sigma a_n e^{-\lambda_n s} = \int_0^\infty s e^{-su} A_\lambda(u)\, du.$$

These formulae are simple examples of a mode of representation of Dirichlet's series by integrals which we shall often have occasion to use.

(2) Every convergent series is summable; more generally, the limits of oscillation of the typical means associated with a series are at most as wide as those of the series itself. When the series is given, it is possible to choose a sequence (λ_n) in such a way that the series shall be summable $(\lambda, 1)$ and have as its sum any number which does not lie outside its limits of oscillation.

(3) When $\lambda_n = e^n$, a series is summable (λ, κ) if and only if it is convergent. It follows from Theorem 21 below that this is true whenever $\lambda_{n+1}/\lambda_n \geqq k > 1$ for all values of n.

(4) The series $\quad 1^{-s} - 2^{-s} + 3^{-s} - \dots$

is summable (n, κ) if and only if $\kappa > -\sigma$. In other words, it is summable by Cesàro's κ-th mean if and only if $\kappa > -\sigma$. For references to proofs of this proposition, when κ is an integer, see the footnote (¶) to p. 20. The general result is distinctly more difficult to prove; a direct proof has been given by Chapman, and another indicated by Hardy and Chapman†.

* The integral may obviously be reduced, by the substitution $\lambda(x) = \xi$, to one in which the exponential factor has the simpler form $e^{-s\xi}$. There is no such fundamental distinction of integrals into types as there is with series. But an integral may be summable (λ, κ) for one form of λ and not for another.

† Chapman, 1; Hardy and Chapman, 1.

Exactly the same result holds for the more general series* $\Sigma e^{ani} n^{-s}$ (provided a is not a multiple of 2π) and for the integral†

$$\int_0^\infty e^{axi} x^{-s} \, dx.$$

(5) The series $\qquad \Sigma e^{Ain^a} n^{-s} \qquad (0<a<1, A \neq 0)$

is summable (n, κ) if $(\kappa+1)a+\sigma>1$.‡

(6) The series $\qquad \Sigma n^{-1-ti} \qquad (t \neq 0)$

is not summable (n, κ) for any value of κ, but is summable $(\log n, \kappa)$ for any positive κ however small§. The series

$$\Sigma n^{-1} (\log n)^{-1-ti}$$

is not summable $(\log n, \kappa)$ for any value of κ, but is summable $(\log\log n, \kappa)$ for any positive κ however small||; and so on generally.

(7) The series Σc_n is summable $(\log n, 1)$ to sum C if and only if

$$\left(C_1+\frac{1}{2}C_2+\ldots+\frac{1}{n}C_n\right)\Big/\log n \rightarrow C.¶$$

(8) If Σa_n is summable (λ, κ) to sum A, and Σb_n is summable (λ, κ) to sum B, then $\Sigma(pa_n+qb_n)$ is summable (λ, κ) to sum $pA+qB$.

(9) If $\Sigma a_n e^{-\lambda_n s}$ is summable (λ, κ) to sum $f(s)$, then

$$a_{m+1} e^{-\lambda_{m+1} s} + a_{m+2} e^{-\lambda_{m+2} s} + \ldots$$

is summable (μ, κ), where $\mu_n = \lambda_{m+n}$, to sum

$$f(s) - a_1 e^{-\lambda_1 s} - \ldots - a_m e^{-\lambda_m s}.$$

(10) If $\Sigma a_n e^{-\lambda_n s}$ is summable (λ, κ) to sum $f(s)$, then $\Sigma a_n e^{-(\lambda_n - \lambda_1)s}$ is summable (μ, κ), where $\mu_n = \lambda_n - \lambda_1$, to sum $e^{\lambda_1 s} f(s)$.

The last two examples will be used later (VII, § 2). The first is a corollary of (8). The second follows from the identity

$$\sum_{\lambda_n - \lambda_1 < \omega} \left(1 - \frac{\lambda_n - \lambda_1}{\omega}\right)^\kappa a_n e^{-(\lambda_n - \lambda_1)s}$$

$$= \left(\frac{\omega+\lambda_1}{\omega}\right)^\kappa e^{\lambda_1 s} \sum_{\lambda_n < \omega+\lambda_1} \left(1 - \frac{\lambda_n}{\omega+\lambda_1}\right)^\kappa a_n e^{-\lambda_n s}.$$

* Chapman, **1**. † Hardy, **6**.

‡ Hardy, **6** (where A is taken to be 1). This paper contains a number of general theorems concerning the relations, from the point of view of convergence and summability, between the series $\Sigma f(n)$ and the integral $\int f(x)\,dx$.

§ Riesz, **1**. See also Hardy, **4**, and Theorems 19, 42, and 47.

|| Hardy, **4**. ¶ Riesz, **2**.

(11) The summability of the series

$$c_1+c_2+c_3+\dots$$

does not, in general, involve that of the series

$$c_2+c_3+c_4+\dots;$$

nor is the converse proposition true. It may even happen that both series are summable and their sums do not differ by c_1. Both propositions are true, however, if the increase of λ_n is sufficiently regular, and in particular if λ_n is a logarithmico-exponential function*, *i.e.* a function defined by any finite combination of logarithms and exponentials.

V

GENERAL ARITHMETIC THEOREMS CONCERNING TYPICAL MEANS

1. The general theorems which we shall prove concerning the summation of series by typical means may be divided roughly into two classes. There are, in the first place, theorems the validity of which does not depend upon any hypothesis that the series considered are Dirichlet's series of any special type. Such theorems we may call 'arithmetic'. There are other theorems in which such a hypothesis is essential. Thus Theorem 23 of Section VI depends upon the fact that we are applying methods of summation of type λ to a series of the same type. Such a theorem we may describe as 'typical' †.

The theorems of this section are all 'arithmetic', those of the following sections mainly (though not entirely) 'typical'.

2. Five lemmas. We shall now give five lemmas which will be useful in the sequel.

LEMMA 4. *If ϕ is a positive function of x such that*

$$\int^{\infty} \phi\, dx$$

is divergent, and $f = o(\phi)$, then

$$\int^{x} f\, dx = o\left(\int^{x} \phi\, dx\right).$$

The proof may be left to the reader.

* Hardy, *Orders of infinity*, p. 17.

† We make this distinction merely for the sake of convenience in exposition and lay no stress upon it.

LEMMA 5. *Let $\phi(x)$, $\psi(x)$ be continuous functions of x such that*

$$\phi(x) \sim Ax^{\alpha}, \quad \psi(x) \sim Bx^{\beta} \qquad (\alpha \geqq 0,\ \beta \geqq 0)$$

as $x \to \infty$. Then

$$\chi(x) = \int_0^x \phi(t)\,\psi(x-t)\,dt \sim AB\frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\,x^{\alpha+\beta+1}.\,*$$

We can write

$$\phi(t) = At^{\alpha} + \phi_1, \quad \psi(t) = Bt^{\beta} + \psi_1,$$

where

$$\phi_1 = o(t^{\alpha}), \quad \psi_1 = o(t^{\beta}).$$

If we substitute these forms of ϕ and ψ in χ, we obtain a sum of four integrals, the first of which gives us

$$AB\int_0^x t^{\alpha}(x-t)^{\beta}\,dt = AB\,\frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\,x^{\alpha+\beta+1}.$$

It remains to prove that the other three integrals are of the form

$$o(x^{\alpha+\beta+1}).$$

Let us take, for example, the integral

$$A\int_0^x t^{\alpha}\,\psi_1(x-t)\,dt = A\int_0^x (x-t)^{\alpha}\,\psi_1(t)\,dt.$$

Given ϵ, we can choose ξ so that

$$|\psi_1(t)| < \epsilon t^{\beta} \qquad (\xi \leqq t \leqq x).$$

Between 0 and ξ, $|\psi_1(t)|$ is less than a number $M(\xi)$ which depends only on ξ. Hence our integral is in absolute value less than

$$|A|\left\{\epsilon\int_0^x (x-t)^{\alpha}\,t^{\beta}\,dt + \xi M(\xi)\,x^{\alpha}\right\},$$

and is therefore of the form $o(x^{\alpha+\beta+1})$. The other integrals may be proved similarly to be also of this form, and thus the proof of the lemma is completed.

LEMMA 6. *If $\kappa > 0$, $\mu > 0$, then*

$$C^{\kappa+\mu}(\omega) = \frac{\Gamma(\kappa+\mu+1)}{\Gamma(\kappa+1)\,\Gamma(\mu)}\int_0^{\omega} C^{\kappa}(u)\,(\omega-u)^{\mu-1}\,du.$$

And if $\kappa > 0$, $\mu < 1$, $\mu \leqq \kappa$, then

$$C^{\kappa-\mu}(\omega) = \frac{\Gamma(\kappa-\mu+1)}{\Gamma(\kappa+1)\,\Gamma(1-\mu)}\int_0^{\omega} \frac{dC^{\kappa}(u)}{du}\,(\omega-u)^{-\mu}\,du.$$

* Chapman, 1. The result is true for $\alpha > -1$, $\beta > -1$: we state it in the form in which we shall use it. By $\phi(x) \sim Ax^{\alpha}$ we mean that $\phi/x^{\alpha} \to A$ as $x \to \infty$.

To prove the first formula, we substitute for $C^\kappa(u)$ its expression as a definite integral*. We thus obtain

$$\int_0^\omega C^\kappa(u)(\omega-u)^{\mu-1}\,du = \kappa\int_0^\omega (\omega-u)^{\mu-1}\,du\int_0^u C(\tau)(u-\tau)^{\kappa-1}\,d\tau$$

$$= \kappa\int_0^\omega C(\tau)\,d\tau\int_\tau^\omega (u-\tau)^{\kappa-1}(\omega-u)^{\mu-1}\,du$$

$$= \frac{\Gamma(\kappa+1)\,\Gamma(\mu)}{\Gamma(\kappa+\mu)}\int_0^\omega C(\tau)(\omega-\tau)^{\kappa+\mu-1}\,d\tau,$$

which is the formula required. The legitimacy of the inversion of the order of integration follows at once from classical theorems †.

When μ is a positive integer, we have the simpler formula

$$C^{\kappa+\mu}(\omega) = (\kappa+1)(\kappa+2)\ldots(\kappa+\mu)\left(\int_0^\omega d\tau\right)^\mu C^\kappa(\tau).$$

To prove the second formula of the lemma, we observe that

$$C^{\kappa-\mu}(\omega) = \frac{1}{\kappa-\mu+1}\frac{d}{d\omega}C^{\kappa-\mu+1}(\omega)$$

$$= \frac{\Gamma(\kappa-\mu+1)}{\Gamma(\kappa+1)\,\Gamma(1-\mu)}\frac{d}{d\omega}\int_0^\omega C^\kappa(u)(\omega-u)^{-\mu}\,du,$$

by the first formula. Integrating by parts, we find

$$\int_0^\omega C^\kappa(u)(\omega-u)^{-\mu}\,du = \frac{1}{1-\mu}\int_0^\omega \frac{dC^\kappa(u)}{du}(\omega-u)^{1-\mu}\,du.$$

Differentiating with respect to ω‡ we obtain the second formula.

LEMMA 7. *If c_n is real, $0 \leqq \xi \leqq \omega$, and $0 < \kappa \leqq 1$, then*

$$|g(\xi,\omega)| = \left|\kappa\int_0^\xi C(\tau)(\omega-\tau)^{\kappa-1}\,d\tau\right| \leqq \operatorname*{Max}_{0\leqq\tau\leqq\xi}|C^\kappa(\tau)|.$$

The truth of the lemma is evident when $\kappa = 1$, and we may therefore suppose $\kappa < 1$. Substituting in the integral which defines $g(\xi,\omega)$ the expression of $C(\tau)$ as an integral given by the second formula of Lemma 6, with $\mu = \kappa$, we obtain

$$g(\xi,\omega) = \frac{\sin\kappa\pi}{\pi}\int_0^\xi (\omega-\tau)^{\kappa-1}\,d\tau\int_0^\tau \frac{dC^\kappa(u)}{du}(\tau-u)^{-\kappa}\,du$$

$$= \int_0^\xi \frac{dC^\kappa(u)}{du}h(u)\,du \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

* It is also easy to prove the lemma by means of the expression of $C^\kappa(u)$ in terms of $c_1, c_2, \ldots$.

† See, for example, de la Vallée-Poussin, *Cours d'analyse infinitésimale*, Vol. 2, ed. 2, Ch. 2.

‡ See the last footnote.

where $$h(u) = \frac{\sin \kappa\pi}{\pi} \int_u^{\xi} (\omega - \tau)^{\kappa - 1} (\tau - u)^{-\kappa} \, d\tau$$
$$= 1 - \frac{\sin \kappa\pi}{\pi} \int_{\xi}^{\omega} (\omega - \tau)^{\kappa - 1} (\tau - u)^{-\kappa} \, d\tau.$$

Now if τ has any fixed value between ξ and ω, and $0 \leqslant u \leqslant \xi$, then $(\tau - u)^{-\kappa}$ increases with u. Hence $h(u)$ is a positive decreasing function of u, always less than 1. Applying the second mean value theorem to the integral (1), we obtain

$$g(\xi, \omega) = h(0)\, C^{\kappa}(\eta) \qquad (0 \leqq \eta \leqq \xi),$$

and this proves the result of the lemma*. In the same way we can prove

LEMMA 8. *If c_n is real, $0 \leqq \xi \leqq \omega$, $\mu > 0$, and $0 < \kappa \leqq 1$, then*

$$\frac{\Gamma(\kappa + \mu + 1)}{\Gamma(\mu + 1)\Gamma(\kappa)} \left| \int_0^{\xi} C^{\mu}(\tau)(\omega - \tau)^{\kappa - 1} d\tau \right| \leqq \operatorname*{Max}_{0 \leqq \tau \leqq \xi} |C^{\kappa + \mu}(\tau)|.$$

We would recommend the reader to pay close attention to Lemmas 7 and 8, and in particular the former. We shall appeal to these lemmas repeatedly in what follows, the part which they play in the theory of integrals of the 'Liouville-Riemann' type being analogous to that played by the classical mean value theorems in the theory of ordinary differential coefficients.

It should be observed that, in the proof of Lemma 7, no appeal is made to the particular structure of the function $C(\tau)$: an analogous result holds for any function which possesses an absolutely convergent integral over any finite interval. Further, the result may be extended to apply to *complex* functions; but as we shall only make use of it when $C(\tau)$ is real, we have given a proof which applies to this case only.

3. First theorem of consistency.

We can now prove

THEOREM 16. *If the series Σc_n is summable (λ, κ) to sum C, it is summable (λ, κ') to the same sum, for every κ' greater than κ.*

Writing $\kappa' = \kappa + \mu$, and applying Lemma 6, we obtain

$$C^{\kappa + \mu}(\omega) = \frac{\Gamma(\mu + \kappa + 1)}{\Gamma(\kappa + 1)\Gamma(\mu)} \int_0^{\omega} C^{\kappa}(u)(\omega - u)^{\mu - 1} du.$$

But $C^{\kappa}(u) \sim Cu^{\kappa}$. Hence the theorem follows immediately from Lemma 5. In particular a convergent series is summable (λ, κ) for all positive values of κ.

* The argument is that used by Riesz, 4.

The following proposition, which we shall have occasion to use later on, is easily established by the same kind of argument: *if*

$$e^{-\rho\omega}A\ (\omega)=O(1),$$

then

$$e^{-(\rho+\delta)\omega}A^{\kappa'}(\omega)=o(1)$$

for any positive δ and any κ' greater than κ.

It is important to observe that *the theorem of consistency holds also for integrals* (IV, § 3). The proof is practically the same as for series.

4. Second theorem of consistency. Theorem 16 states a relation between methods of summation of the same type λ and of different orders κ, κ'. There is a much deeper theorem which concerns methods of the same order but of different types.

THEOREM 17. *If the series Σc_n is summable (l, κ), where $l_n=e^{\lambda_n}$, then it is summable (λ, κ) to the same sum.*

The proof of this theorem is somewhat intricate, and we shall confine ourselves, for the sake of simplicity, to two cases, viz. (i) that in which κ is *integral*, (ii) that in which $0<\kappa<1$.* These are the cases of greatest interest; and this course is one which we shall adopt in regard to a number of the theorems which follow. Further, it is easy to see that we may without loss of generality suppose C, the sum of the series, to be zero: this can always be secured by an alteration in the first term of the series.

We are given that

$$\int_1^w C_l(t)\,(w-t)^{\kappa-1}\,dt=o\,(w^{\kappa})\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and we have to prove that

$$\int_0^\omega C_\lambda(\tau)\,(\omega-\tau)^{\kappa-1}\,d\tau=o\,(\omega^{\kappa})\quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

If we put $\omega=\log w$, $\tau=\log t$, and observe that $C_\lambda(\log t)=C_l(t)$, we see that (2) may be written in the form

$$\int_1^w C_l(t)\,(\log w-\log t)^{\kappa-1}\frac{dt}{t}=o\,(\log w)^{\kappa}\quad\ldots\ldots\ldots(3).$$

* We shall indicate summarily (see § 7) the lines of the proof in the most general case.

5. (*a*) *Proof when κ is an integer.* In this case

$$C_l(t) = \frac{1}{\kappa!}\left(\frac{d}{dt}\right)^{\kappa} C_l^{\kappa}(t).$$

We substitute this expression for $C_l(t)$ in (3), and integrate κ times by parts. We can then show that both the terms integrated out and the integral which remains are of the form required.

In the first place, all the integrated terms vanish except one, which arises from the last integration by parts and is a constant multiple of

$$C_l^{\kappa}(w)\left[\left(\frac{d}{dt}\right)^{\kappa-1}\frac{(\log w - \log t)^{\kappa-1}}{t}\right]_{t=w}$$

$$= (-1)^{\kappa-1}(\kappa-1)!\, w^{-\kappa} C_l^{\kappa}(w) = o(1).$$

Thus we need only consider the residuary integral, which is a constant multiple of

$$\int_1^w C_l^{\kappa}(t)\left(\frac{d}{dt}\right)^{\kappa}\frac{(\log w - \log t)^{\kappa-1}}{t}\,dt \text{ ....(4).}$$

Now it is easily verified that

$$\left(\frac{d}{dt}\right)^{\kappa}\frac{(\log w - \log t)^{\kappa-1}}{t} = t^{-\kappa-1}\Sigma H_{r,s}(\log w)^r(\log t)^s \text{ ...(5),}$$

where $H_{r,s}$ is a constant, and

$$r + s \leqq \kappa - 1 \text{ ....(6).}$$

But, using Lemma 4 and the inequality (6), we have

$$(\log w)^r \int_1^w C_l^{\kappa}(t)(\log t)^s t^{-\kappa-1}\,dt = (\log w)^r \int_1^w (\log t)^s o\left(\frac{1}{t}\right)dt$$

$$= o(\log w)^{r+s+1} = o(\log w)^{\kappa}.$$

Hence the integral (4) is of the form required, and the theorem is proved when κ is an integer.

6. (*b*) *Proof when* $0 < \kappa < 1$. In this proof we shall suppose the c's real. There is plainly no loss of generality involved in this hypothesis, as we can consider the real and imaginary parts of the series separately.

We have again to establish equation (3) of § 4. By Theorem 16, $C_l^1(t) = o(t)$. Hence we can choose ν so that

$$|C_l^1(t)| < \epsilon t \qquad (t \geqq \nu),$$

and evidently we may suppose $\nu > 1$. We then choose a value of w

greater than 3ν, and denote by M the upper limit of $C_l^1(t)$ in $(1, \nu)$; and we write the integral (3) of § 4 in the form

$$\int_1^\nu + \int_\nu^{w/3} + \int_{w/3}^w = J_1 + J_2 + J_3.$$

In the first place

$$|J_1| < M\nu\left(\log\frac{w}{\nu}\right)^{\kappa-1} = o(\log w)^\kappa \text{ .... (1).}$$

Secondly, integrating by parts, we obtain

$$J_2 = \frac{3}{w}(\log 3)^{\kappa-1} C_l^1(\tfrac{1}{3}w) - \frac{1}{\nu}\left(\log\frac{w}{\nu}\right)^{\kappa-1} C_l^1(\nu)$$
$$+ \int_\nu^{w/3} C_l^1(t)\left\{(\kappa-1)\left(\log\frac{w}{t}\right)^{\kappa-2} + \left(\log\frac{w}{t}\right)^{\kappa-1}\right\}\frac{dt}{t^2}.$$

The first two terms are in absolute value less than a constant multiple of ϵ, and the last than*

$$\epsilon\int_\nu^{w/3}\left(\log\frac{w}{t}\right)^{\kappa-1}\frac{dt}{t} < \frac{\epsilon}{\kappa}\left(\log\frac{w}{\nu}\right)^\kappa < \frac{\epsilon}{\kappa}(\log w)^\kappa.$$

Hence, for sufficiently large values of w, we have

$$|J_2| < \frac{2\epsilon}{\kappa}(\log w)^\kappa \text{ .... (2).}$$

Finally, by the second mean value theorem,

$$J_3 = \frac{3}{w}\int_{w/3}^{\xi} C_l(t)\left(\log\frac{w}{t}\right)^{\kappa-1} dt$$
$$= \frac{3}{w}\int_{w/3}^{\xi} C_l(t)(w-t)^{\kappa-1}\left(\frac{\log w - \log t}{w-t}\right)^{\kappa-1} dt,$$

where $\frac{1}{3}w \leqq \xi \leqq w$. Now it will easily be verified that the function

$$\left(\frac{\log w - \log t}{w-t}\right)^{\kappa-1}$$

increases steadily from $t=1$ to $t=w$, and that its limit when $t \to w$ is $w^{1-\kappa}$. Hence, using the second mean value theorem again, we obtain

$$|J_3| = \frac{3}{w}\left(\frac{\log w - \log\xi}{w-\xi}\right)^{\kappa-1}\left|\int_{\xi_1}^{\xi} C_l(t)(w-t)^{\kappa-1} dt\right|$$
$$\leqq 3w^{-\kappa}\left|\int_{\xi_1}^{\xi} C_l(t)(w-t)^{\kappa-1} dt\right|,$$

* Here we use the facts that $0 < 1-\kappa < 1$ and that, as $\log 3 > 1$,

$$\left(\log\frac{w}{t}\right)^{\kappa-2} < \left(\log\frac{w}{t}\right)^{\kappa-1}$$

for $\nu < t < \frac{1}{3}w$.

where $\frac{1}{2}w \leqq \xi_1 \leqq \xi \leqq w$. But, by Lemma 7,

$$\left| \int_{\xi_1}^{\xi} C_l(t)\,(w-t)^{\kappa-1}\,dt \right| \leqq \frac{2}{\kappa} \operatorname*{Max}_{\frac{1}{2}w \leqq t \leqq w} |C_l^{\kappa}(t)|.$$

Hence, as $C_l^{\kappa}(t) = o(t^{\kappa})$, we have

$$J_3 = o(1) = o(\log w)^{\kappa} \quad\text{.......................(3).}$$

From (1), (2), and (3) the result of the theorem follows.

7. We have thus proved Theorem 17 when κ is an integer and when $0 < \kappa < 1$. If κ is non-integral, but greater than 1, it is necessary to combine our two methods of demonstration. We write $\mathbf{k} = [\kappa]$, and integrate the integral (3) of § 4 by parts until we have replaced $C_l(t)$ by $C_l^{\mathbf{k}}(t)$, which then plays in the proof the part played by $C_l(t)$ in § 6.

A particularly interesting special case of Theorem 17 is

THEOREM 18. *If a series is summable by arithmetic means, it is summable by logarithmic means of the same order**.

That the converse is not true is shown by the first example of IV, § 4 (6). The examples there given suggest as a general conclusion that *the efficacy of the method* (λ, κ) *increases as the rate of increase of the function* λ *decreases*. This general idea may be made more precise by the following theorem, which includes Theorem 17 as a special case, and may be established by reasoning of the same character.

THEOREM 19. *Let* μ *be any logarithmico-exponential function of* λ, *which tends to infinity with* λ, *but more slowly than* λ. *Then, if the series* Σc_n *is summable* (λ, κ), *it is summable* (μ, κ)†.

Thus, if we imagine λ as running through the functions of the logarithmico-exponential scale of infinity, such as e^n, n, $\log n$, $\log\log n$, ...‡, we obtain a sequence of systems of methods (λ, κ) of gradually increasing efficacy.

8. THEOREM 20. *If* $\lambda_1 > 0$, *and* Σc_n *is summable* (λ, κ), *then* $\Sigma c_n \lambda_n^{-\kappa}$ *is summable* (l, κ).

This theorem is interesting as a companion to Theorem 17. Its proof is very similar, though slightly more complicated. We shall suppose as before that $C = 0$.

* This theorem was published without proof by Riesz, **2**.

† We can in reality say rather more, viz. that summability (λ, κ) implies summability (μ, κ) if $\mu = O(\lambda^{\Delta})$, where Δ is any constant however large. A special case of this theorem has been proved by Berwald, **1**.

‡ The result of IV, § 4, (3) shows that it is useless to consider types higher than e^n.

We are given that

$$\kappa \int_{\lambda_1}^{\omega} C_\lambda(\tau)(\omega-\tau)^{\kappa-1}\, d\tau = o(\omega^\kappa) \quad \text{.....................(1)};$$

and we have to show that

$$\kappa w^{-\kappa}\int_{l_1}^{w} D_l(t)(w-t)^{\kappa-1}\, dt = \kappa e^{-\kappa\omega}\int_{\lambda_1}^{\omega} D_\lambda(\tau)(e^\omega - e^\tau)^{\kappa-1} e^\tau\, d\tau \quad \text{...(2)}$$

tends to a limit as $\omega \to \infty$. Here $D_l(t)$, $D_\lambda(\tau)$ denote sum-functions formed from the series Σd_n, where $d_n = c_n \lambda_n^{-\kappa}$. It will easily be verified that

$$D_\lambda(\tau) = \kappa \int_{\lambda_1}^{\tau} C_\lambda(u)\frac{du}{u^{\kappa+1}} + \frac{C_\lambda(\tau)}{\tau^\kappa}.$$

We substitute this expression for $D_\lambda(\tau)$ in (2), and so obtain

$$\kappa^2 e^{-\kappa\omega}\int_{\lambda_1}^{\omega}(e^\omega - e^\tau)^{\kappa-1} e^\tau\, d\tau \int_{\lambda_1}^{\tau} C_\lambda(u)\frac{du}{u^{\kappa+1}} + \kappa e^{-\kappa\omega}\int_{\lambda_1}^{\omega}(e^\omega - e^\tau)^{\kappa-1} e^\tau\, C_\lambda(\tau)\frac{d\tau}{\tau^\kappa}$$

......(3).

The first term, when we invert the order of integration, and perform the integration with respect to τ, becomes

$$\kappa e^{-\kappa\omega}\int_{\lambda_1}^{\omega} C_\lambda(u)(e^\omega - e^u)^\kappa \frac{du}{u^{\kappa+1}}.$$

(*a*) We suppose first that κ is an integer. We integrate κ times by parts, as in the proof of Theorem 17. All the integrated terms vanish, so that we obtain

$$\frac{(-1)^\kappa e^{-\kappa\omega}}{(\kappa-1)!}\int_{\lambda_1}^{\omega} C_\lambda^\kappa(u)\left(\frac{d}{du}\right)^\kappa \left\{\frac{(e^\omega - e^u)^\kappa}{u^{\kappa+1}}\right\} du \quad \text{..............(4)}.$$

It will be easily verified that the differential coefficient may be expressed in the form

$$(-1)^\kappa(\kappa+1)(\kappa+2)\ldots 2\kappa\, u^{-2\kappa-1} e^{\kappa\omega} + \Sigma\, G_{r,s}\, e^{r\omega} e^{(\kappa-r)u} u^{-s} \quad \text{......(5)},$$

where $G_{r,s}$ is a constant, and

$$\kappa - r \geqq 1, \quad s - \kappa \geqq 1 \quad \text{............................(6)}.$$

If we substitute this expression in (4), and observe that

$$u^{-s} C_\lambda^\kappa(u) = o(u^{\kappa-s}) = o(1),$$

we find that the coefficient of $G_{r,s}$ is

$$e^{-\omega(\kappa-r)}\int_{\lambda_1}^{\omega} o\{e^{(\kappa-r)u}\}\, du = o(1).$$

Hence all these terms may be neglected, and it appears that the expression (4) tends, as $\omega \to \infty$, to the limit

$$\frac{2\kappa!}{\kappa!(\kappa-1)!}\int_{\lambda_1}^{\infty} C_\lambda^\kappa(u)\frac{du}{u^{2\kappa+1}} \quad \text{..........................(7)}.$$

There remains the second term of (3), which may be discussed in the same manner. In this case, when we perform the integration by parts, there is one integrated term which does not vanish, as in § 5. It is however easily seen that this term has the limit zero*. The integral which remains can then be divided into a number of parts all of which can be shown to tend to zero by an argument practically identical with that employed above. Thus our final conclusion is that $\Sigma c_n \lambda_n^{-\kappa}$ is summable (l, κ), and that its sum is given by the integral (7).

We have supposed $C=0$. In order to extend our result to the general case, we have only to show that the sum of the series is given by (7) in the particular case when $c_1 = C$, $c_2 = c_3 = \ldots = 0$. This we may leave as an exercise for the reader. Finally we may observe that the theorem gives us the maximum of information possible. This may be seen by considering the case in which $\lambda_n = n$, $l_n = e^n$, $\kappa = 1$. Then summability (l, κ) is equivalent to convergence, and the theorem asserts that *if* Σc_n *is summable by Cesàro's first mean, then* $\Sigma (c_n/n)$ *is convergent.* In this proposition the factor $1/n$ cannot be replaced by any factor which tends more slowly to zero.

(*b*) Suppose next that $0 < \kappa < 1$. As in § 6, we suppose the c's real. We have again to show that the expression (2) tends to a limit as $\omega \to \infty$. By Theorem 16, $C_\lambda^1(\tau) = o(\tau)$; and as

$$\kappa \int_{\lambda_1}^{\omega} C_\lambda(\tau) \frac{d\tau}{\tau^{\kappa+1}} = \frac{\kappa C_\lambda^1(\omega)}{\omega^{\kappa+1}} + \kappa(\kappa+1) \int_{\lambda_1}^{\omega} \frac{C_\lambda^1(\tau)}{\tau^{\kappa+2}} d\tau,$$

it follows that the integral

$$I = \kappa \int_{\lambda_1}^{\infty} C_\lambda(\tau) \frac{d\tau}{\tau^{\kappa+1}}$$

is convergent. And from this it follows, by the analogue of Theorem 16 for integrals, that

$$\kappa e^{-\kappa\omega} \int_{\lambda_1}^{\omega} C_\lambda(\tau) (e^\omega - e^\tau)^\kappa \frac{d\tau}{\tau^{\kappa+1}} \to I$$

as $\omega \to \infty$.

It remains only to show that the second term of (3) tends to zero. We separate it into two parts corresponding to the ranges of integration $(\lambda_1, 1)$, $(1, \omega)$†; and it is evident that the first part tends to zero. The second may be written in the form.

$$J = \kappa e^{-\kappa\omega} \int_1^{\omega} \chi(\tau) C_\lambda(\tau) (\omega - \tau)^{\kappa-1} d\tau,$$

where

$$\chi(\tau) = \left(\frac{e^\omega - e^\tau}{\omega - \tau}\right)^{\kappa-1} \frac{e^\tau}{\tau^\kappa}.$$

Now it may easily be verified that, as τ increases from 1 to ω, $\chi(\tau)$ increases towards the limit $\omega^{-\kappa} e^{\kappa\omega}$. Hence, by the second mean value theorem, we obtain

$$J = \kappa \omega^{-\kappa} \int_{\xi}^{\omega} C_\lambda(\tau) (\omega - \tau)^{\kappa-1} d\tau, \qquad (0 \leqq \xi \leqq \omega);$$

* It is owing to the presence of this term that the series $\Sigma c_n \lambda_n^{-\kappa}$ is not necessarily summable (λ, κ') for any κ' less than κ.

† If $\lambda_1 \geqq 1$, this is unnecessary.

and it follows at once from Lemma 7 and the equation (1) that the limit of J is zero.

We leave it as an exercise for the reader to show that

$$I = \frac{\Gamma(2\kappa+1)}{\Gamma(\kappa)\Gamma(\kappa+1)} \int_0^\infty C_\lambda^\kappa(u) \frac{du}{u^{2\kappa+1}}.$$

9. Theorem 21. *If Σc_n is summable (λ, κ) to sum C, then*

$$C_n - C = o\left(\frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_n}\right)^\kappa.$$

The proof of this theorem is extremely simple when κ is integral. In the equation

$$C^\kappa(\omega) = \sum_{\lambda_n \leqq \omega} c_n (\omega - \lambda_n)^\kappa = C\omega^\kappa + o(\omega^\kappa)$$

we write $$\omega = \lambda_n,\ \lambda_n + h,\ \lambda_n + 2h,\ \ldots,\ \lambda_n + \kappa h,$$

where $\kappa h = \lambda_{n+1} - \lambda_n$, and form the κ-th difference of the $\kappa + 1$ equations thus obtained. Since the κ-th difference of $(\omega - \lambda_n)^\kappa$ is a numerical multiple of h^κ, the result of the theorem follows at once.

Now let us suppose that $0 < \kappa < 1$; and let us assume that the c_n's are real and $C = 0$, as evidently we may do without loss of generality. Then

$$\int_0^{\lambda_{n+1}} (\lambda_{n+1} - \tau)^{\kappa-1} C(\tau)\, d\tau = o(\lambda_{n+1}{}^\kappa).$$

By Lemma 7, we have also

$$\int_0^{\lambda_n} (\lambda_{n+1} - \tau)^{\kappa-1} C(\tau)\, d\tau = o(\lambda_{n+1}{}^\kappa);$$

and so the same is true of the integral taken between the limits λ_n and λ_{n+1}. But $C(\tau) = C_n$ for $\lambda_n < \tau < \lambda_{n+1}$, and so

$$(\lambda_{n+1} - \lambda_n)^\kappa C_n = o(\lambda_{n+1}{}^\kappa),$$

which proves the theorem*.

A more general theorem is

Theorem 22. *Suppose that Σc_n is summable (λ, κ) to sum C, and that $0 < \kappa' \leqq \kappa$, $\lambda_n \leqq \omega \leqq \lambda_{n+1}$. Then*

$$C^{\kappa'}(\omega) - C\omega^{\kappa'} = o\left\{\frac{\lambda_n{}^\kappa}{(\lambda_n - \lambda_{n-1})^{\kappa-\kappa'}} + \frac{\lambda_{n+1}{}^\kappa}{(\lambda_{n+1} - \lambda_n)^{\kappa-\kappa'}}\right\} \quad \ldots\ldots(1).$$

If κ' is integral, then we may write simply

$$C^{\kappa'}(\omega) - C\omega^{\kappa'} = o\left\{\frac{\lambda_{n+1}{}^\kappa}{(\lambda_{n+1} - \lambda_n)^{\kappa-\kappa'}}\right\} \quad \ldots\ldots\ldots\ldots\ldots(2);$$

and this result holds for $\kappa' = 0$, provided $\lambda_n < \omega$.†

* The proof for the case in which κ is non-integral and greater than 1 is contained in that of Theorem 22.

† This distinction arises from the fact that $C(\omega)$ is discontinuous for $\omega = \lambda_n$.

We shall as usual take $C=0$, and assume that c_n is real. First suppose that κ' is an integer, and let us write $\mathbf{k}=[\kappa]$.* Let us also write

$$\Omega_1=\lambda_n+h,\ \ldots,\ \Omega_{\mathbf{k}}=\lambda_n+\mathbf{k}h,\ \Omega_{\mathbf{k}+1}=\lambda_{n+1},$$

where
$$h=(\lambda_{n+1}-\lambda_n)/(\mathbf{k}+1).$$

If Ω denotes any one of these numbers, we have, by Lemma 6,

$$\frac{\Gamma(\kappa+1)}{\Gamma(\mathbf{k}+1)\Gamma(\kappa-\mathbf{k})}\int_0^\Omega C^{\mathbf{k}}(\tau)(\Omega-\tau)^{\kappa-\mathbf{k}-1}\,d\tau=C^\kappa(\Omega).$$

Using Lemma 8 and the equations $C^\kappa(\Omega)=o(\Omega^\kappa)=o(\lambda_{n+1}{}^\kappa)$, we obtain

$$\int_{\lambda_n}^\Omega C^{\mathbf{k}}(\tau)(\Omega-\tau)^{\kappa-\mathbf{k}-1}\,d\tau=o(\lambda_{n+1}{}^\kappa) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Integrating $\mathbf{k}$ times by parts, and observing that $C(\tau)$ is constant in the range of integration, we can express the integral in (3) as the sum of constant multiples of the $\mathbf{k}+1$ functions

$$(\lambda_{n+1}-\lambda_n)^{\kappa-\mathbf{k}}C^{\mathbf{k}}(\lambda_n),\ (\lambda_{n+1}-\lambda_n)^{\kappa-\mathbf{k}+1}C^{\mathbf{k}-1}(\lambda_n),$$
$$\ldots,\ (\lambda_{n+1}-\lambda_n)^\kappa C(\lambda_n+0)\ldots\ldots\ldots(4)\dagger.$$

This process leads, for the $\mathbf{k}+1$ different values of Ω, to $\mathbf{k}+1$ different linear combinations of the functions (4), each of which is of the form $o(\lambda_{n+1}{}^\kappa)$. But it is easy to verify that these $\mathbf{k}+1$ linear combinations are linearly independent, the determinant of the system being a 'Vandermonde-Cauchy'‡ determinant, different from zero; and so the functions themselves are of this form.

The last function is $(\lambda_{n+1}-\lambda_n)^\kappa C(\omega)$, and so the theorem is proved for $\kappa'=0$. The last function but one is $(\lambda_{n+1}-\lambda_n)^{\kappa-1}C^1(\lambda_n)$; and so $C^1(\lambda_n)$ is of the form prescribed by the theorem. Hence

$$C^1(\omega)=C^1(\lambda_n)+\int_{\lambda_n}^\omega C(\tau)\,d\tau=C^1(\lambda_n)+\int_{\lambda_n}^\omega o\left\{\frac{\lambda_{n+1}{}^\kappa}{(\lambda_{n+1}-\lambda_n)^\kappa}\right\}d\tau$$
$$=o\left\{\frac{\lambda_{n+1}{}^\kappa}{(\lambda_{n+1}-\lambda_n)^{\kappa-1}}\right\}.$$

Hence the theorem is proved for $\kappa'=1$. Repeating the argument we establish it for $\kappa'=0, 1, 2, \ldots, \mathbf{k}$.

We pass now to the case in which κ' is not integral, and we write $\mathbf{k}'=[\kappa']$.§

* We shall in this case give a complete proof for all values of κ, integral or not. The method of proof is that used by Riesz, **4**, in proving the equivalence of the 'arithmetic' means with Cesàro's.

† In the last function we must write λ_n+0 and not λ_n, on account of the discontinuity of $C(\tau)$.

‡ See Pascal, *I Determinanti* (Manuali Hoepli), pp. 166 *et seq.*

§ It is very curious that the simpler result which holds when κ' is integral should not hold always; but it is possible to show by examples that this is so.

By Lemma 6, we have

$$C^{\kappa'}(\omega)=\frac{\Gamma(\kappa'+1)}{\Gamma(\mathbf{k}'+1)\,\Gamma(\kappa'-\mathbf{k}')}\int_0^{\omega} C^{\mathbf{k}'}(\tau)\,(\omega-\tau)^{\kappa'-\mathbf{k}'-1}\,d\tau$$

$$=\frac{\Gamma(\kappa'+1)}{\Gamma(\mathbf{k}'+1)\,\Gamma(\kappa'-\mathbf{k}')}\left(\int_0^{\lambda_{n-1}}+\int_{\lambda_{n-1}}^{\omega}\right)=J_1+J_2,$$

say. We begin by considering J_2. Dividing the interval (λ_{n-1}, ω) into the two parts $(\lambda_{n-1}, \lambda_n)$, (λ_n, ω), and using the result (2) for $C^{\mathbf{k}'}(\tau)$, we find

$$J_2=o\left\{\frac{\lambda_n{}^{\kappa}}{(\lambda_n-\lambda_{n-1})^{\kappa-\mathbf{k}'}}\right\}\int_{\lambda_{n-1}}^{\lambda_n}(\omega-\tau)^{\kappa'-\mathbf{k}'-1}\,d\tau$$

$$+o\left\{\frac{\lambda_{n+1}{}^{\kappa}}{(\lambda_{n+1}-\lambda_n)^{\kappa-\mathbf{k}'}}\right\}\int_{\lambda_n}^{\omega}(\omega-\tau)^{\kappa'-\mathbf{k}'-1}\,d\tau.$$

Now $\kappa'-\mathbf{k}'-1$ is negative and $\kappa'-\mathbf{k}'$ positive. Hence

$$\int_{\lambda_{n-1}}^{\lambda_n}(\omega-\tau)^{\kappa'-\mathbf{k}'-1}\,d\tau\leqq\int_{\lambda_{n-1}}^{\lambda_n}(\lambda_n-\tau)^{\kappa'-\mathbf{k}'-1}\,d\tau=\frac{(\lambda_n-\lambda_{n-1})^{\kappa'-\mathbf{k}'}}{\kappa'-\mathbf{k}'},$$

$$\int_{\lambda_n}^{\omega}(\omega-\tau)^{\kappa'-\mathbf{k}'-1}\,d\tau=\frac{(\omega-\lambda_n)^{\kappa'-\mathbf{k}'}}{\kappa'-\mathbf{k}'}\leqq\frac{(\lambda_{n+1}-\lambda_n)^{\kappa'-\mathbf{k}'}}{\kappa'-\mathbf{k}'}.$$

Thus

$$J_2=o\left\{\frac{\lambda_n{}^{\kappa}}{(\lambda_n-\lambda_{n-1})^{\kappa-\kappa'}}\right\}+o\left\{\frac{\lambda_{n+1}{}^{\kappa}}{(\lambda_{n+1}-\lambda_n)^{\kappa-\kappa'}}\right\}\quad\text{...............(5).}$$

In order to obtain an upper limit for J_1, we integrate $\mathbf{k}-\mathbf{k}'$ times by parts. We find that

$$J_1=\sum_{\mu=1}^{\mathbf{k}-\mathbf{k}'}\frac{\Gamma(\kappa'+1)}{\Gamma(\mathbf{k}'+\mu+1)\,\Gamma(\kappa'-\mathbf{k}'-\mu+1)}(\omega-\lambda_{n-1})^{\kappa'-\mathbf{k}'-\mu}\,C^{\mathbf{k}'+\mu}(\lambda_{n-1})$$

$$+\frac{\Gamma(\kappa'+1)}{\Gamma(\mathbf{k}+1)\,\Gamma(\kappa'-\mathbf{k})}\int_0^{\lambda_{n-1}}C^{\mathbf{k}}(\tau)\,(\omega-\tau)^{\kappa'-\mathbf{k}-1}\,d\tau.$$

Since $\kappa'-\mathbf{k}'-\mu<0$, we may replace the powers of $\omega-\lambda_{n-1}$ in the first line by the corresponding powers of $\lambda_n-\lambda_{n-1}$. If we do this, and at the same time apply the result (2) to the factors $C^{\mathbf{k}'+\mu}(\lambda_{n-1})$, we find at once that every term in the first line is of the form

$$o\left\{\frac{\lambda_n{}^{\kappa}}{(\lambda_n-\lambda_{n-1})^{\kappa-\kappa'}}\right\}\quad\text{..............................(6).}$$

On the other hand, the integral which occurs in the second line may, by the second mean value theorem, be expressed in the form

$$(\omega-\lambda_{n-1})^{\kappa'-\kappa}\int_{\xi}^{\lambda_{n-1}}C^{\mathbf{k}}(\tau)\,(\omega-\tau)^{\kappa-\mathbf{k}-1}\,d\tau,$$

where $0\leqq\xi\leqq\lambda_{n-1}$. Replacing ω by λ_n in the external factor, and applying Lemma 8 to the integral, in the same way in which we used Lemma 7 in the proof of Theorem 21, we see that this part of J_1 is also of the form (6). The proof of Theorem 22 is thus completed.

In the particularly important case in which $\lambda_n=n$, Theorem 22 shows that *if the series* Σc_n *is summable* (n, κ), *and* $0\leqq\kappa'<\kappa$, *then*

$$C^{\kappa'}(\omega)=o(\omega^{\kappa}).$$

VI

ABELIAN AND TAUBERIAN THEOREMS

1. Generalisations of Theorem 2 and its corollaries. We pass now to an important theorem which occupies the same place in the theory of the summability of Dirichlet's series as does Theorem 2 in the elementary theory of their convergence. But first we shall prove a subsidiary proposition which will be useful to us.

LEMMA 9. *If $f(u)$ is integrable, and $f(u)=o(u^\kappa)$, then*

$$\int_0^\omega s^{\kappa+1} e^{-su} f(u)\,du$$

tends to a limit as $\omega \rightarrow \infty$, uniformly for all values of s in the angle **a** *defined by $|\operatorname{am} s| \leqq \alpha < \frac{1}{2}\pi$.*

Choose ξ so that

$$|f(u)| < \epsilon u^\kappa \qquad (u \geqq \xi).$$

Then

$$\left|\int_{\omega_1}^{\omega_2} s^{\kappa+1} e^{-su} f(u)\,du\right| < \epsilon \int_0^\infty |s|^{\kappa+1} e^{-\sigma u} u^\kappa\,du$$

$$= \epsilon\,\Gamma(\kappa+1)\left(\frac{|s|}{\sigma}\right)^{\kappa+1} \leqq \epsilon\,\Gamma(\kappa+1)(\sec\alpha)^{\kappa+1},$$

if only $\omega_2 > \omega_1 \geqq \xi$. As ξ is independent of the position of s in the angle, the lemma is proved*.

2. THEOREM 23. *If Σa_n is summable (λ, κ), then $\Sigma a_n e^{-\lambda_n s}$ is uniformly summable throughout the angle* **a**.

THEOREM 24. *The sum of the series $\Sigma a_n e^{-\lambda_n s}$ is equal to*

$$\frac{1}{\Gamma(\kappa+1)}\int_0^\infty s^{\kappa+1} e^{-s\tau} A^\kappa(\tau)\,d\tau$$

at all points of **a** *other than the origin; and to*

$$A + \frac{1}{\Gamma(\kappa+1)}\int_0^\infty s^{\kappa+1} e^{-s\tau}\{A^\kappa(\tau) - A\tau^\kappa\}\,d\tau,$$

where A is the sum of Σa_n, at all points of **a**.†

* The presence of the factor $s^{\kappa+1}$ is of course essential for the truth of the result.

† Compare IV, § 4, (1) for the simplest case of such a representation of a Dirichlet's series by an integral.

We shall, as usual, prove this in the cases in which (i) κ is integral and (ii) $0<\kappa<1$. We observe first that, if $c_n = a_n e^{-\lambda_n s}$, and λ_p denotes the last λ less than ω, then

$$C^\kappa(\omega) = \sum_{\lambda_n<\omega} c_n(\omega-\lambda_n)^\kappa = \sum_1^{p-1} A_n \Delta\{e^{-\lambda_n s}(\omega-\lambda_n)^\kappa\} + A_p e^{-\lambda_p s}(\omega-\lambda_p)^\kappa$$

$$= -\int_0^\omega A(\tau)\frac{d}{d\tau}\{e^{-s\tau}(\omega-\tau)^\kappa\}\,d\tau \quad \ldots\ldots\ldots\ldots(1).$$

(*a*) ***Proof when*** κ ***is integral.*** We suppose first that

$$A=0,\quad A^\kappa(\omega)=o(\omega^\kappa).$$

Integrate (1) κ times by parts. All the integrated terms vanish save one, which is

$$e^{-s\omega}A^\kappa(\omega)=o(\omega^\kappa) \quad \ldots\ldots\ldots\ldots(2),$$

uniformly throughout $\mathbf{a}$. Thus we need only consider the integral remaining over, which, when divided by ω^κ, is

$$\frac{(-1)^{\kappa+1}\omega^{-\kappa}}{\kappa\,!}\int_0^\omega A^\kappa(\tau)\left(\frac{d}{d\tau}\right)^{\kappa+1}\{e^{-s\tau}(\omega-\tau)^\kappa\}\,d\tau \quad \ldots\ldots\ldots(3).$$

Now it will easily be verified that

$$\left(\frac{d}{d\tau}\right)^{\kappa+1}\{e^{-s\tau}(\omega-\tau)^\kappa\} = (-1)^{\kappa+1}e^{-s\tau}s^{\kappa+1}\omega^\kappa + e^{-s\tau}\Sigma H_{i,j,k}\,s^i\omega^j\tau^k,$$

where $H_{i,j,k}$ is a constant, and

$$i=j+k+1,\quad j\leqq\kappa-1 \quad \ldots\ldots\ldots\ldots(4).$$

When we substitute this expression in (3), the first term, which we may call the *principal* term, gives rise to the integral

$$\frac{1}{\kappa\,!}\int_0^\omega s^{\kappa+1}e^{-s\tau}A^\kappa(\tau)\,d\tau \quad \ldots\ldots\ldots\ldots(5).$$

As $\omega\to\infty$, this integral, by Lemma 9, tends uniformly to a limit, viz. the first integral of Theorem 24. Thus, when $A=0$, all that is necessary to complete the proof is to show that

$$\omega^{-\kappa+j}\int_0^\omega s^i\tau^k e^{-s\tau}A^\kappa(\tau)\,d\tau = o(1),$$

uniformly in $\mathbf{a}$; i, j, k being subject to the inequalities (4). We divide this integral into the two parts

$$\omega^{-\kappa+j}\int_0^\nu + \omega^{-\kappa+j}\int_\nu^\omega = J_1+J_2,$$

choosing ν so that $|A^\kappa(\tau)|<\epsilon\tau^\kappa$ throughout the range of integration in J_2. Further we observe that there is a constant M such that $|A^\kappa(\tau)|<M\tau^\kappa$ for all values of τ.

The function

$$F(x) = x^{-\kappa+j} \int_0^x e^{-\tau} \tau^{k+\kappa} \, d\tau$$

has (for positive values of x) a maximum μ.* And

$$|J_2| < \epsilon \omega^{-\kappa+j} |s|^i \int_0^\omega e^{-\sigma\tau} \tau^{k+\kappa} \, d\tau$$

$$= \epsilon \omega^{-\kappa+j} \left(\frac{|s|}{\sigma}\right)^i \sigma^{i-k-\kappa-1} \int_0^{\sigma\omega} e^{-y} y^{k+\kappa} \, dy$$

$$= \epsilon (|s|/\sigma)^i F(\sigma\omega) \dagger \leqq \epsilon\mu (\sec a)^i \quad \text{...........(6).}$$

Also

$$|J_1| < M\omega^{-\kappa+j} |s|^i \int_0^\nu e^{-\sigma\tau} \tau^{k+\kappa} \, d\tau$$

$$= M\omega^{-\kappa+j} (|s|/\sigma)^i \nu^{\kappa-j} F(\nu\sigma)$$

$$\leqq M\mu (\sec a)^i (\nu/\omega)^{\kappa-j} \quad \text{......................(7).}$$

From (6) and (7) it follows that, by taking first ν and then ω sufficiently large, we can make $J_1 + J_2$ as small as we please. This completes the proof of Theorems 23 and 24, when κ is an integer and $A = 0$.

Now suppose $A \neq 0$. Then the series

$$(-A + a_1 e^{-\lambda_1 s}) + a_2 e^{-\lambda_2 s} + a_3 e^{-\lambda_3 s} + \ldots$$

is, in virtue of what has just been proved, uniformly summable in **a**, with sum

$$\frac{1}{\kappa!} \int_0^\infty s^{\kappa+1} e^{-s\tau} \{A^\kappa(\tau) - A\tau^\kappa\} \, d\tau. \ddagger$$

And the series

$$A + 0 + 0 + \ldots$$

is, as may be seen at once by actual formation of the typical means, or inferred from the theorem of consistency, also uniformly summable in **a**. Moreover its sum is A, which, *except when* $s = 0$, is equal to

$$\frac{A}{\kappa!} \int_0^\infty s^{\kappa+1} e^{-s\tau} \tau^\kappa \, d\tau.$$

Combining these results we obtain Theorems 23 and 24. It should be observed that, when $A \neq 0$, the first integral in Theorem 24 is *not* uniformly convergent, or even continuous for $s = 0$.

* Since $-\kappa + j + k + \kappa + 1 = j + k + 1 = i > 0$, the function has the limit 0 when $x \to 0$ as well as when $x \to \infty$.

† Since $i - k - \kappa - 1 = -\kappa + j$.

‡ The sum function of the modified series is equal to $A(\tau) - A$.

3. (*b*) *Proof when* $0<\kappa<1$. As before, we begin by supposing $A=0$. We have, by (1) of § 2, to discuss the limit of

$$-\omega^{-\kappa}\int_0^\omega A(\tau)\frac{d}{d\tau}\{e^{-s\tau}(\omega-\tau)^\kappa\}\,d\tau=\kappa\omega^{-\kappa}e^{-s\omega}\int_0^\omega A(\tau)(\omega-\tau)^{\kappa-1}\,d\tau$$
$$-\omega^{-\kappa}\int_0^\omega A(\tau)\frac{d}{d\tau}\{(e^{-s\tau}-e^{-s\omega})(\omega-\tau)^\kappa\}\,d\tau.$$

The first term on the right-hand side is

$$e^{-s\omega}\omega^{-\kappa}A^\kappa(\omega)=o(1)$$

uniformly in **a**. Thus we need only consider the second term, which, when we integrate by parts, takes the form

$$\omega^{-\kappa}\int_0^\omega A^1(\tau)\frac{d^2}{d\tau^2}\{(e^{-s\tau}-e^{-s\omega})(\omega-\tau)^\kappa\}\,d\tau=J_1+J_2+J_3\quad\ldots(1),$$

where J_1, J_2, J_3 are three integrals containing, under the sign of integration, the function $A^1(\tau)$ multiplied respectively by

$$\left.\begin{array}{ll}\text{(i)} & s^2e^{-s\tau}(\omega-\tau)^\kappa,\\ \text{(ii)} & 2\kappa s e^{-s\tau}(\omega-\tau)^{\kappa-1},\\ \text{(iii)} & \kappa(\kappa-1)(e^{-s\tau}-e^{-s\omega})(\omega-\tau)^{\kappa-2}\end{array}\right\}\quad\ldots\ldots\ldots\ldots(2).$$

In the first place

$$J_1=\omega^{-\kappa}\int_0^\omega s^2e^{-s\tau}A^1(\tau)(\omega-\tau)^\kappa\,d\tau\quad\ldots\ldots\ldots\ldots\ldots(3).$$

Now Σa_n is summable $(\lambda, 1)$, and so $A^1(\tau)=o(\tau)$. Hence, by Lemma 9, the integral

$$\int_0^\omega s^2e^{-s\tau}A^1(\tau)\,d\tau$$

tends to a limit, as $\omega\to\infty$, uniformly in **a**. And hence, by the analogue for integrals* of Theorem 16, the integral (3) does the same. Further, the value of the limit is

$$\int_0^\infty s^2e^{-s\tau}A^1(\tau)\,d\tau=\frac{s^2}{\Gamma(1+\kappa)\Gamma(1-\kappa)}\int_0^\infty e^{-s\tau}\,d\tau\int_0^\tau A^\kappa(u)\frac{du}{(\tau-u)^\kappa}$$
$$=\frac{s^2}{\Gamma(1+\kappa)\Gamma(1-\kappa)}\int_0^\infty A^\kappa(u)\,du\int_u^\infty e^{-s\tau}\frac{d\tau}{(\tau-u)^\kappa}$$
$$=\frac{1}{\Gamma(1+\kappa)}\int_0^\infty s^{\kappa+1}e^{-su}A^\kappa(u)\,du.\dagger$$

* See the end of V, § 3.

† In the first line we use Lemma 6. The inversion of the order of integration presents no difficulty, all the integrals concerned being absolutely convergent.

It remains to prove that J_2 and J_3 tend uniformly to zero. We write

$$J_2 = 2\kappa\omega^{-\kappa}\int_0^\omega se^{-s\tau}(\omega-\tau)^{\kappa-1}A^1(\tau)\,d\tau = 2\kappa\omega^{-\kappa}\left(\int_0^\nu + \int_\nu^\omega\right) = J_2' + J_2'',$$

say. We can choose ν so that $|A^1(\tau)| < \epsilon\tau$ for $\tau > \nu$, and we can suppose $\omega - \nu > 1$. Further, there is a constant M such that $|A^1(\tau)| < M\tau$ for all positive values of τ. Also $\tau e^{-\tau} < 1$ for all values of τ, and $|s|/\sigma \leqq \sec\alpha$ throughout the angle **a**. Hence, denoting by K the constant $2\kappa \sec\alpha$, we have

$$|J_2'| < KM\omega^{-\kappa}\int_0^\nu \sigma\tau e^{-\sigma\tau}(\omega-\tau)^{\kappa-1}\,d\tau < KM\nu\omega^{-\kappa} \quad \text{......(4).}$$

Similarly we have

$$|J_2''| < K\epsilon\omega^{-\kappa}\int_\nu^\omega \sigma\tau e^{-\sigma\tau}(\omega-\tau)^{\kappa-1}\,d\tau < \frac{K\epsilon}{\kappa} \quad \text{.........(5).}$$

From (4) and (5) it follows that we can make J_2 as small as we please, uniformly throughout **a**, by making first ν and then ω sufficiently large.

In order to discuss J_3 we observe that, by Lemma 2,

$$|e^{-s\tau} - e^{-s\omega}| \leqq (e^{-\sigma\tau} - e^{-\sigma\omega})\sec\alpha < (\omega-\tau)\,\sigma e^{-\sigma\tau}\sec\alpha.$$

The discussion is then almost exactly the same as in the case of J_2. The proof of the theorems is thus completed.

4. Lines of summability. Analytic character of the sum. From Theorem 23 we can at once deduce a series of important corollaries, analogous to those deduced from Theorem 2 in II, §§ 2 *et seq.*

THEOREM 25. *If the series is summable* (λ, κ)* *for a value of* s *whose real part is* σ, *then it is summable* (λ, κ) *for all values of* s *whose real part is greater than* σ.

THEOREM 26. *There is a number* σ_κ *such that the series is summable when* $\sigma > \sigma_\kappa$ *and not summable when* $\sigma < \sigma_\kappa$. *We may have* $\sigma_\kappa = -\infty$ *or* $\sigma_\kappa = \infty$.†

We now define the abscissa σ_κ, the line $\sigma = \sigma_\kappa$, and the half-plane $\sigma > \sigma_\kappa$ of summability (λ, κ), just as we did in II, § 2 when $\kappa = 0$. It is evident (from the first theorem of consistency) that

$$\bar{\sigma} \geqq \sigma_0 \geqq \sigma_1 \geqq \sigma_2 \geqq \dots .$$

* It should be added that the result of Theorem 25 remains true if we assume only that Σa_n is *finite* (λ, κ): cf. the first footnote to p. 4. The first representation of the sum as an integral is also valid in this case, as may easily be shown by a trifling modification of the proof of Theorem 24.

† See p. 4 for an explanation of the meaning to be attached to this phrase.

THEOREM 27. *If D is any finite region for all points of which $\sigma \geqq \sigma_\kappa + \delta > \sigma_\kappa$, then the series is uniformly summable (λ, κ) throughout D, and its sum represents a branch $f(s)$ of an analytic function regular throughout D. Further, the series*

$$\Sigma\, a_n \lambda_n^{\rho} e^{-\lambda_n s},$$

where ρ is any number real or complex, and λ_n^{ρ} has its principal value, is also uniformly summable (λ, κ) throughout D, and, when ρ is a positive integer, represents the function

$$(-1)^{\rho} f^{(\rho)}(s).$$

The proof of this theorem is similar to that of Theorem 4. One additional remark is however necessary. When we prove that the summability of Σa_n involves that of

$$\Sigma a_n \lambda_n^{\rho} = \Sigma a_n e^{\rho \log \lambda_n},$$

whenever the real part of ρ is negative, we must appeal, not to Theorem 23, but to its analogue Theorem 29 below; for the means (λ, κ) are the means of the second kind for the series $\Sigma a_n \lambda_n^{-s}$.

THEOREM 28. *If the series is summable (λ, κ) for $s = s_0$, and has the sum $f(s_0)$, then $f(s) \rightarrow f(s_0)$ when $s \rightarrow s_0$ along any path lying entirely inside the angle whose vertex is at s_0 and which is similar and similarly situated to the angle* **a**.

There is also an obvious generalisation of Theorem 6 which we shall not state at length.

5. Summability by typical means of the second kind. In §§ 1—4 we have considered exclusively typical means of the first kind. All the results of these sections, however, remain true when we work with means of the second kind, except that Theorems 23 and 24 must be replaced by

THEOREM 29. *If the series Σa_n is summable (l, κ), then the series $\Sigma a_n l_n^{-s}$ is uniformly summable (l, κ) in the angle* **a**. *Its sum is (except for $s=0$) equal to the integral*

$$\frac{\Gamma(s+\kappa+1)}{\Gamma(\kappa+1)\Gamma(s)} \int_1^{\infty} A_l^{\kappa}(u)\, u^{-s-\kappa-1}\, du.*$$

It is not necessary that we should do more than indicate the lines of the proof when $0<\kappa<1$. The κ-th mean formed from $\Sigma a_n l_n^{-s}$ may be expressed, by the same transformation as was used at the beginning of § 2, in the form

$$-w^{-\kappa} \int_1^{w} A_l(t) \frac{d}{dt} \{t^{-s}(w-t)^{\kappa}\}\, dt.$$

Arguing as at the beginning of § 3, we replace this expression by

$$w^{-\kappa} \int_1^{w} A_l^1(t) \frac{d^2}{dt^2} \{(t^{-s} - w^{-s})(w-t)^{\kappa}\}\, dt.$$

* As $l_1 = e^{\lambda_1} \geqq 1$, and $A_l^{\kappa}(u) = 0$ for $u \leqq l_1$, the lower limit may be 0, 1, or l_1 indifferently.

Finally, as $w \to \infty$, this tends uniformly to the limit

$$s(s+1)\int_1^\infty A_l^1(t)\,t^{-s-2}\,dt = \frac{s(s+1)}{\Gamma(1+\kappa)\,\Gamma(1-\kappa)}\int_1^\infty t^{-s-2}\,dt\int_1^t A_l^\kappa(u)\,\frac{du}{(t-u)^\kappa}$$

$$= \frac{s(s+1)}{\Gamma(1+\kappa)\,\Gamma(1-\kappa)}\int_1^\infty A_l^\kappa(u)\,du\int_u^\infty \frac{t^{-s-2}}{(t-u)^\kappa}\,dt$$

$$= \frac{\Gamma(s+\kappa+1)}{\Gamma(\kappa+1)\,\Gamma(s)}\int_1^\infty A_l^\kappa(u)\,u^{-s-\kappa-1}\,du.$$

We add one more theorem.

THEOREM 30. *The lines of summability are the same for the means of the first and the second kind.*

We shall content ourselves with sketching the proof of this theorem. In the first place, if the series is summable (l, κ) for $s=s_0$, it is, by Theorem 17, summable (λ, κ) for $s=s_0$, and *a fortiori* for $\sigma>\sigma_0$. On the other hand, if it is summable (λ, κ) for $s=s_0$, the series

$$\Sigma\, a_n \lambda_n^{-p} e^{-\lambda_n s},$$

where p is any integer greater than κ, is, by Theorem 20, summable (l, κ) for $s=s_0$, and *a fortiori* for $\sigma>\sigma_0$. Hence, by the analogue of Theorem 27 for means of the second kind, the original series is summable (l, κ) for $\sigma>\sigma_0$.*

6. Explicit formulae for σ_κ. The actual values of the abscissae of summability are given by the following generalisation of Theorem 8.

THEOREM 31. *The abscissa of summability σ_κ, if positive, is given by*

$$\sigma_\kappa = \overline{\lim}\,\frac{\log|A_\lambda^\kappa(\omega)|}{\omega} = \overline{\lim}\,\frac{\log|A_l^\kappa(w)|}{\log w} - \kappa.$$

The proof of these results follows the general lines of that of Theorems 23 and 24, but is easier, as no question of uniformity is involved. As the proof is not very interesting in itself, we shall confine ourselves to indicating the general line of the argument for means of the first kind.

We assume first that

$$A^\kappa(\tau) = o\{e^{\tau(\eta+\delta)}\} \quad \text{..............................(1)}$$

for a definite positive η and every positive δ. If now we follow the argument of §§ 2, 3, we can show without difficulty that the series is summable (λ, κ) if $\sigma>\eta$, and that its sum is the first integral of Theorem 24. If on the other hand the series is summable (λ, κ) when $s=\eta+\delta$, and

$$c_n = a_n e^{-\lambda_n s}, \quad a_n = c_n e^{\lambda_n s},$$

we have obviously

$$C^\kappa(\tau) = o(e^{\epsilon\tau}),$$

* It is also possible to give a direct proof of this theorem similar to, but rather easier than, that of Theorem 20. We have to prove that the summability (λ, κ) of $\Sigma\, a_n e^{-\lambda_n s}$ involves the summability (l, κ) of $\Sigma\, a_n e^{-\lambda_n (s+\delta)}$ for any positive δ.

for every positive ϵ. Hence, performing the same arguments with $-s$ in the place of s, we deduce (1) with $\delta+\epsilon$ in the place of δ. It follows that if (1) holds for η, but for no smaller number than η, then the series is summable when $\sigma>\eta$ but not when $\sigma<\eta$. This proves the first equation in Theorem 31.

7. Tauberian Theorems. In this section we shall state a number of theorems whose general character is 'Tauberian'; that is to say, which are developments of an idea which appeared first in Tauber's well-known 'converse of Abel's Theorem'*. In spite of the great intrinsic interest of these theorems we omit the proofs, as we shall not have occasion to make any applications of the results.

THEOREM 32†. *If*

$$\text{(i)}\quad a_n = o\{(\lambda_n - \lambda_{n-1})/\lambda_n\}$$

and (ii) *the series* $\sum a_n e^{-\lambda_n s}$, *then certainly convergent for* $\sigma > 0$, *tends to a limit* A *as* $s \to 0$ *through positive values, then the series* $\sum a_n$ *is convergent and has the sum* A.

THEOREM 33‡. *The conclusion of Theorem* 32 *still holds if the condition* (i) *is replaced by the more general condition*

$$\text{(i}'\text{)}\quad \lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n = o(\lambda_n).$$

Moreover the conditions (i′) *and* (ii) *are necessary and sufficient for the convergence of the series* $\sum a_n$.

THEOREM 34§. *If* $\lambda_n - \lambda_{n-1} = o(\lambda_n)$, *then the condition* (i) *of Theorem* 32 *may be replaced by the more general condition*

$$a_n = O\{(\lambda_n - \lambda_{n-1})/\lambda_n\}.$$

THEOREM 35‖. *If*

$$\text{(i)}\quad a_n = O\{(\lambda_n - \lambda_{n-1})/\lambda_n\}$$

and (ii) $\sum a_n$ *is summable* (λ, κ) *to sum* A, *then* $\sum a_n$ *is convergent to sum* A.

THEOREM 36. *If*

$$\lambda_n = O(\lambda_n - \lambda_{n-1}),$$

then no series can be summable (λ, κ) *unless it is convergent.*

The last theorem is an immediate consequence of Theorem 21. It contains as a particular case the result of IV, § 4, (3); viz. that the

* Tauber, **1**. For a general explanation of the character of a 'Tauberian' theorem see Hardy and Littlewood, **1**.

† Landau, **8**. ‡ Schnee, **8**. § Littlewood, **1**. See also Landau, **11**, **12**.

‖ Hardy, **8**. If $\lambda_n - \lambda_{n-1} = o(\lambda_n)$, this may be deduced as a corollary from Theorems 28 and 34. See also Hardy, **4**.

means (e^n, κ) are 'trivial' in the sense that no non-convergent series is summable by means of them.

The theorems of this section are capable of many interesting generalisations for which we must refer elsewhere*. We add however one important theorem which resembles Theorems 32—36 in that its conditions include a condition as to the order or average order of the coefficient a_n, but differs from them fundamentally in that it depends on the theory of functions of a complex variable.

THEOREM 37†. *If*

$$\text{(i)}\quad A_n = a_1 + a_2 + \dots + a_n = o(e^{\lambda_n c}) \qquad (c \geqq 0)$$

and (ii) *the series* $\Sigma a_n e^{-\lambda_n s}$, *then certainly convergent for* $\sigma > c$, *represents a function* $f(s)$ *regular for* $s = s_0 = c + t_0 i$, *then the series is convergent for* $s = s_0$, *and its sum is* $f(s_0)$.

It should be observed that (i) is certainly satisfied if $c > 0$ and

$$\text{(i}'\text{)}\quad a_n = o\{(\lambda_n - \lambda_{n-1})\, e^{\lambda_{n-1} c}\}.$$

This is no longer true if $c = 0$. But it is easy to see, by applying a linear transformation to the variable s, that the theorem obtained by putting $c = 0$ in (i′), viz. '*if*

$$a_n = o(\lambda_n - \lambda_{n-1})$$

then the series $\Sigma a_n e^{-\lambda_n s}$ *is convergent at every regular point of the line* $\sigma = 0$' is certainly true in all cases in which $\lambda_n - \lambda_{n-1} = O(1)$. This theorem is the direct generalisation of a well-known theorem of Fatou ‡, to which it reduces when $\lambda_n = n$. It should also be observed that Fatou's theorem and its extension become false when $O(\lambda_n - \lambda_{n-1})$ is substituted for $o(\lambda_n - \lambda_{n-1})$.

8. **Examples to illustrate §§ 4—7.** (1) For the series Σn^{-s}, we have

$$\bar{\sigma} = \sigma_0 = \sigma_1 = \sigma_2 = \dots = 1.$$

See IV, § 4, (6).

(2) For the series $\Sigma(-1)^n n^{-s}$, we have $\sigma_\kappa = -\kappa$. See IV, § 4, (4).

(3) For the series $\Sigma e^{Ain^a} n^{-s}$, where $0 < a < 1$ and $A \neq 0$, we have $\sigma_\kappa = 1 - (\kappa + 1)\, a$. See IV, § 4, (5).

* See in particular Landau, **8**; Hardy and Littlewood, **2**, **4**, **5**. In reference to the original theorem of Tauber see Pringsheim, **2**, **3**; Bromwich, *Infinite series*, p. 251.

† Riesz, **4**. The proof of the general theorem is still unpublished. For the case $\lambda_n = n$ see Riesz, **5**; for the case $\lambda_n = \log n$ see Landau, **8**. The condition (i) is a necessary condition for the existence of any points of convergence on the line $\sigma = c$ (Jensen, **2**).

‡ Fatou, **1**; Riesz, **5**. The latter paper contains a number of further theorems of a similar character.

(4) Each of the series (2) and (3) is summable, by typical means of *some* order, all over the plane, and consequently represents an integral function of s. It is of some interest to obtain an example of a series which represents an integral function, but cannot, for some values of s, be summed by any typical mean. Such an example is afforded by the series

$$\Sigma e^{i(\log n)^2} n^{-s}.$$

Here all the lines of summability coincide in the line $\sigma = 1$. None the less the series represents an integral function. So does the series

$$\Sigma (-1)^n e^{-sn^a} \qquad (0 < a < 1),$$

all of whose lines of summability coincide in the line $\sigma = 0$.

(5) For the series

$$1^{-s} - 2^{-s} + 4^{-s} - 5^{-s} + 27^{-s} - 28^{-s} + \ldots,$$

in which $a_n = 1$ if $n = m^m$, $a_n = -1$ if $n = m^m + 1$, and $a_n = 0$ otherwise, we have

$$\bar{\sigma} = \sigma_0 = 0, \quad \sigma_\kappa = -\kappa.*$$

(6) The series $$\Sigma \frac{(-1)^n n^a}{(\log n)^s} \qquad (a > 0)$$

is summable (λ, κ), where $\kappa > a$, for all values of s, but is never summable (λ, κ) if $\kappa < a$. It is summable (λ, a) if $\sigma \geqq -a$, and summable (l, a) if $\sigma > -a$.

(7) The series $$\Sigma \frac{n^{-1-ai} (\log n)^\beta}{(\log \log n)^s} \qquad (a \neq 0, \ \beta > 0)$$

is summable everywhere by typical means of the first or second kind (or indeed by ordinary logarithmic means) of order greater than β, but is never summable by arithmetic means of any order.

* Bohr, 2.

VII

FURTHER DEVELOPMENTS OF THE THEORY OF FUNCTIONS REPRESENTED BY DIRICHLET'S SERIES

1. We shall now use the idea of summation by typical means to obtain generalisations of some of the most important theorems of Section III.

THEOREM 38. *Suppose the series* $\sum a_n e^{-\lambda_n s}$ *summable or finite* (λ, κ) *for* $s=\beta$. *Then*

$$f(s)=o(|t|^{\kappa+1})$$

uniformly for $\sigma \geqq \beta+\epsilon>\beta$.

We may plainly suppose, without loss of generality, that $\beta=0$. There is a constant M such that

$$|A^\kappa(\tau)|<M\tau^\kappa$$

for all positive values of τ. Now, by Theorem 24,

$$f(s)=s^{\kappa+1}\int_{\lambda_1}^{\infty} A^\kappa(\tau)e^{-s\tau}d\tau$$

for $\sigma>0$. Hence

$$|f(s)|<M|s|^{\kappa+1}\int_{\lambda_1}^{\infty}\tau^\kappa e^{-\sigma\tau}d\tau=M\left(\frac{|s|}{\sigma}\right)^{\kappa+1}\int_0^\infty v^\kappa e^{-v}dv$$

$$\leqq M(\sec a)^{\kappa+1}\Gamma(\kappa+1)=O(1)=o(|t|^{\kappa+1}),$$

uniformly in any angle of the type **a** of Lemma 9. Hence, in proving the theorem, we may confine ourselves to the parts of the half-plane $\sigma\geqq\epsilon$ which lie outside this angle; and so we may suppose $|s/t|$ less than a constant cosec a. This being so, we have

$$f(s)=s^{\kappa+1}\left(\int_{\lambda_1}^{\nu}+\int_{\nu}^{\infty}\right)A^\kappa(\tau)e^{-s\tau}d\tau=J_1+J_2,$$

say.

Now
$$|J_2|<M|s|^{\kappa+1}\int_\nu^\infty \tau^\kappa e^{-\sigma\tau}d\tau$$

$$\leqslant M(\operatorname{cosec} a)^{\kappa+1}|t|^{\kappa+1}\int_\nu^\infty \tau^\kappa e^{-\epsilon\tau}d\tau;$$

and so if δ is any positive number, we have $|J_2|<\delta|t|^{\kappa+1}$ for all values of ν greater than a number ν_0 which depends on ϵ and δ but not on s. When ν_0 has been chosen so that this inequality may hold, we have, by integration by parts,

$$J_1 = -s^\kappa e^{-s\nu} A^\kappa(\nu) + s^\kappa \int_{\lambda_1}^{\nu} e^{-s\tau} \frac{dA^\kappa(\tau)}{d\tau} d\tau \, *;$$

and so, since $|e^{-s\tau}| < 1$,

$$|J_1| < H(\nu) |s|^\kappa \leqq H(\nu) (\operatorname{cosec} \alpha)^\kappa |t|^\kappa,$$

where $H(\nu)$ depends on ν alone. Accordingly

$$|f(s)| < H(\nu)(\operatorname{cosec} \alpha)^\kappa |t|^\kappa + \delta |t|^{\kappa+1} < 2\delta |t|^{\kappa+1},$$

if $|t|$ is large enough. Thus the theorem is proved†.

2. Generalisation of Theorem 13. We proceed next to a generalisation of Perron's formula discussed in § 2 of Section III.

THEOREM 39. *If the series is summable* (λ, κ), *where* $\kappa > 0$, *for* $s = \beta$, *and* $c > 0$, $c > \beta$, *then*

$$\frac{1}{\Gamma(\kappa+1)} \sum_{\lambda_n < \omega} a_n (\omega - \lambda_n)^\kappa = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f(s)}{s^{\kappa+1}} e^{\omega s} ds \quad \ldots\ldots(1).$$

This theorem depends on a generalisation of Lemma 3.

LEMMA 10. *We have*

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{us}}{s^{\kappa+1}} ds \begin{cases} = \dfrac{u^\kappa}{\Gamma(\kappa+1)} & (u \geqq 0), \\ = 0 & (u \leqq 0), \end{cases}$$

c and κ *being positive.*

We leave the verification of this lemma to the reader. It may be deduced without difficulty, by means of Cauchy's Theorem, from Hankel's expression of the reciprocal of the Gamma-function as a contour integral‡.

Let us suppose§ that $\lambda_m < \omega < \lambda_{m+1}$, and write

$$g(s) = e^{\omega s} \left\{ f(s) - \sum_1^m a_n e^{-\lambda_n s} \right\} = \sum_{m+1}^{\infty} a_n e^{-(\lambda_n - \omega)s}.$$

Then what we have to prove reduces, in virtue of the lemma, to showing that

$$\int_{c-i\infty}^{c+i\infty} \frac{g(s)}{s^{\kappa+1}} ds = 0.$$

* The subject of integration may (if $\kappa < 1$) have isolated infinities across which it is absolutely integrable, but the integration by parts is permissible in any case.

† The result of the theorem is true, *a fortiori*, if (l, κ) be substituted for (λ, κ). It was given in this form, for integral values of κ, and for ordinary Dirichlet's series, by Riesz, **1**, and Bohr, **2**, **5**.

‡ Hankel, **1**; see also Heine, **1**, and Whittaker and Watson, *Modern Analysis* (ed. 2), p. 238. For a proof of results equivalent to those of Lemma 10, without the use of Cauchy's Theorem, see Dirichlet's *Vorlesungen über die Lehre von den einfachen und mehrfachen bestimmten Integralen* (ed. Arendt), pp. 166 *et seq.* The formulae may, in substance, be traced back to Cauchy.

§ Cf. III, § 2.

We have $$g(s) = e^{-(\lambda_{m+1} - \omega)s} h(s),$$
where $$h(s) = a_{m+1} + a_{m+2} e^{-(\lambda_{m+2} - \lambda_{m+1})s} + \dots.$$
This series is summable (μ, κ), where $\mu_n = \lambda_{m+n} - \lambda_{m+1}$, for $s = \beta$.* Hence $h(s) = o(|t|^{\kappa+1})$, uniformly for $\sigma \geqq c$. This relation replaces the equation $h(s) = o(|t|)$ used in the proof of Theorem 13; and the proof of Theorem 39 now follows exactly the same lines as that of the latter theorem. The final formula is valid even when $\omega = \lambda_n$, as the left-hand side is a continuous function of ω, and the integral is uniformly convergent.

More generally we have

$$\frac{1}{\Gamma(\kappa+1)} \sum_{\lambda_n < \omega} a_n e^{-\lambda_n s_0} (\omega - \lambda_n)^\kappa = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f(s)}{(s-s_0)^{\kappa+1}} e^{\omega(s-s_0)} \, ds \quad \dots(2),$$

if $c > \sigma_0$, $c > \beta$.

It is important for later applications to observe that the range of validity of the formulae (1) and (2) may be considerably extended. Let us suppose only that the series is summable (λ, κ) for *some* values of s, say for $\sigma > d$, and that the function $f(s)$ thus defined is regular for $\sigma > \beta$, where $\beta < d$, and satisfies the equation

$$f(s) = o(|t|^{\kappa+1}) \quad \dots\dots\dots\dots(3)$$

uniformly for $\sigma \geqq \beta + \epsilon > \beta$, however small ϵ may be. Then the theorem tells us that

$$\frac{1}{\Gamma(\kappa+1)} \sum_{\lambda_n < \omega} a_n (\omega - \lambda_n)^\kappa = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{f(s)}{s^{\kappa+1}} e^{\omega s} \, ds$$

if $\gamma > 0$, $\gamma > d$. But, applying Cauchy's Theorem to the rectangle formed by the points on the lines $\sigma = c$, $\sigma = \gamma$ whose ordinates are $-T_1$ and T_2, and observing that, in virtue of (3), the contributions of the sides of the rectangle parallel to the real axis tend to zero when T_1 and T_2 tend to infinity, we see that the equation (1) still holds. A similar extension may be given to (2).

3. **Analogous formulae for means of the second kind.** There is a companion theorem to Theorem 39, viz.

THEOREM 40. *If* $\sum a_n e^{-\lambda_n s} = \sum a_n l_n^{-s}$ *is summable* (l, κ) *for* $s = \beta$, *and* $c > 0$, $c > \beta$, *then*

$$w^{-\kappa} \sum_{l_n < w} a_n (w - l_n)^\kappa = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s) \frac{\Gamma(\kappa+1)\Gamma(s)}{\Gamma(\kappa+1+s)} w^s \, ds \quad \dots\dots(1).\dagger$$

* See IV, § 4, (9), (10).

† The quotient of Γ-functions which figures under the sign of integration reduces, when κ is integral, to the κ-th difference of $1/s$.

As Theorem 39 depends on Lemma 10, so Theorem 40 depends upon

LEMMA 11. *If* $c>0$ *then*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\kappa+1)\Gamma(s)}{\Gamma(\kappa+1+s)}v^s\,ds=\left(1-\frac{1}{v}\right)^\kappa \quad (v\geqq 1),$$
$$=0 \quad (v\leqq 1).$$

If we write
$$(1-x)^\kappa=\sum_0^\infty B_r^\kappa x^r$$
we have
$$\frac{\Gamma(\kappa+1)\Gamma(s)}{\Gamma(\kappa+1+s)}=\int_0^1 x^{s-1}(1-x)^\kappa\,dx=\sum_0^\infty\frac{B_r^\kappa}{s+r} \quad \text{..............(2).}$$

If we observe that
$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{v^s}{s+r}\,ds=v^{-r}\ (v>1),\quad =\tfrac{1}{2}\ (v=1)^*,\quad =0\ (v\leqq 1),$$
we see that the result of the lemma follows by substituting the series (2) under the sign of integration and integrating term by term. The details of the proofs of the lemma, and then of the theorem, present no particular difficulty, and we content ourselves with indicating the necessary formulae.

There is a generalisation of (1) corresponding to (2) of § 2, viz.
$$w^{-\kappa}\sum_{l_n<w}a_n l_n^{-s_0}(w-l_n)^\kappa=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}f(s)\frac{\Gamma(\kappa+1)\Gamma(s-s_0)}{\Gamma(\kappa+1+s-s_0)}w^{s-s_0}\,ds\ldots(3),$$
where $c>\sigma_0$, $c>\beta$. Both of the formulae (1) and (3), established originally on the hypothesis that $\sum a_n l_n^{-s}$ is summable (l, κ) for $s=\beta$, may then be extended to the case in which the series is only known to be summable (l, κ) for *some* values of s, and $f(s)$ satisfies the conditions stated at the end of § 2.

4. We are now in a position to consider an important group of theorems which differ fundamentally in character from those which we have considered hitherto. In such theorems as, for example, 23, 24, 27, 29, or 38, we start from the assumption that our series is summable for some particular value of s, and deduce properties of the function represented by the sum of the series. We shall now have to deal with theorems in which, to put the matter roughly, properties of the series are deduced from those of the function†.

One preliminary remark is necessary. When we speak of 'the function' we mean, of course, 'the function defined by means of, or associated with, the series.' That is to say, we imply that, for *some* values of s at any rate, *some* method of summation can be applied to

* If $v=1$ the principal value of the integral (in the sense explained in III, § 2) must be taken.

† The classical example of such a theorem is Taylor's Theorem, as proved by Cauchy for functions of a complex variable.

the series so as to give rise to the function. It is obviously, for our present purposes, the natural course to suppose that *for sufficiently large values of* σ, *say for* $\sigma > d$, *the series is summable by typical means of sufficiently high order.* There is thus an analytic function $f(s)$ associated with the series, and possibly capable of analytical continuation outside the known domain of summability of the series. In the theorems which follow we suppose that this is the case, and assume certain additional properties of $f(s)$. We then deduce from these properties more precise information as to the summability of the series.

5. THEOREM 41*. *Suppose that* $f(s)$ *is regular for* $\sigma > \eta$, *where* $\eta < d$. *Suppose further that* κ *and* κ' *are positive numbers such that* $\kappa' < \kappa$, *and that, however small be* δ,

$$f(s) = O(|s|^{\kappa'})$$

uniformly for $\sigma \geqq \eta + \delta > \eta$. *Then* $f(s)$ *is summable* (l, κ), *and a fortiori summable* (λ, κ), *for* $\sigma > \eta$.

If the series $\Sigma a_n l_n^{-s}$ is, for any values of s, summable (λ, κ), we know, by Theorem 40 and its extensions given at the end of § 3, that

$$w^{-\kappa} \sum_{l_n < w} a_n l_n^{-s_0} (w - l_n)^\kappa = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s) H(s - s_0) w^{s-s_0} ds \ldots (1),$$

where H is a certain product of Gamma-functions, provided only $c > \sigma_0$ and $c > \eta$. We can however free ourselves in this case from the assumption of the existence of a half-plane of summability (λ, κ). The series is summable (λ, k) somewhere, for *some* value of k, and therefore, if m is a sufficiently large positive integer, somewhere summable $(\lambda, \kappa + m)$. Hence we deduce the formula (1), with $\kappa + m$ in the place of κ. Now it will easily be verified that if we multiply (1) by w^κ, differentiate with respect to w, and divide by $\kappa w^{\kappa-1}$, we obtain a formula which differs from (1) only in the substitution of $\kappa - 1$ for κ. Hence, by m differentiations, we can pass from $\kappa + m$ to κ. That the process of differentiation under the integral sign is legitimate follows at once from the relations

$$\frac{\Gamma(\kappa + p + 1)\Gamma(s - s_0)}{\Gamma(\kappa + p + 1 + s - s_0)} = O(|t|^{-\kappa-p-1}), \quad f(s) = o(|t|^{\kappa'}),$$

where $\kappa' < \kappa$ and $p = 0, 1, \ldots, m-1$; for the integrals obtained by differentiation are all absolutely and uniformly convergent.

* For $\lambda_n = \log n$, Riesz, **1**: in the general case, Riesz, **2**. Theorem 2 of the latter note includes Theorem 41 in virtue of Theorem 30.

Suppose now that $\eta < \sigma_0 < c$. Choose a number γ such that $\gamma > \eta$ and $\sigma_0 - 1 < \gamma < \sigma_0$, as is obviously possible. Then between the lines $\sigma = \gamma$, $\sigma = c$ lies one pole of $H(s - s_0)$, viz. $s = s_0$, with residue 1. Hence, by a simple application of Cauchy's Theorem, we obtain*

$$w^{-\kappa} \sum_{l_n < w} a_n l_n^{-s_0} (w - l_n)^\kappa - f(s_0) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} f(s)\, w^{s - s_0} H(s - s_0)\, ds \ldots (2).$$

But it is easy to see that the modulus of the integral is less than a constant multiple of $w^{\gamma - \sigma_0}$, and so tends to zero. Thus the theorem is established.

We add some remarks which will be of importance in the sequel. Let us suppose that $f(s)$ is *bounded* in every half-plane $\sigma \geqq \eta + \delta > \eta$. Then, if $\gamma = \sigma_0 - \theta$, where $0 < \theta < 1$, we have, for values of s situated on the line $\sigma = \gamma$,

$$|H(s - s_0)| < K |s - s_0|^{-\kappa - 1} = K \{(t - t_0)^2 + \theta^2\}^{-\frac{1}{2}(\kappa + 1)},$$

where K denotes a number which depends on κ and θ but not on σ_0 or t_0. Hence it follows that, throughout the domain $\sigma_0 \geqq \eta + \delta > \eta$, the integral on the right-hand side of (2) is less than a constant multiple of

$$w^{\gamma - \sigma_0} \int_{-\infty}^{\infty} \frac{dt}{\{(t - t_0)^2 + \theta^2\}^{\frac{1}{2}(\kappa + 1)}}.$$

This integral has obviously a value independent of t_0.† Hence it follows that *if $f(s)$ is limited in every half-plane $\sigma \geqq \eta + \delta > \eta$, the series is uniformly summable (l, κ) in every such half-plane, for any assigned positive value of κ.*

The same remarks apply as regards summability (λ, κ): they are not, as is the mere assertion of simple summability, immediate corollaries of the corresponding remarks concerning summability (l, κ); but it would be easy to complete Theorem 17 in such a way that they would become so. As we shall only make use of these remarks in the case of means of the second kind, it will not be necessary for us to go into this point in detail.

6. With Theorem 41 must be associated the following two more precise theorems.

* We apply Cauchy's Theorem to a rectangle whose shorter ends are made to tend to infinity. Since $f(s) = O(|s|^{\kappa'})$,

$$H(s - s_0) = O(|s|^{-\kappa - 1}),$$

and $\kappa' < \kappa$, the contributions of these ends tend to zero.

† It is important to observe that the argument would fail at this point if κ' were not zero.

THEOREM 42*. *If $f(s)$ is regular for $\sigma \geqq \eta$, except that it has, on the line $\sigma = \eta$, a finite number of poles or algebraical infinities of order less than $\kappa+1$; if further*

$$f(s) = O(|s|^{\kappa'}),$$

where $0 \leqslant \kappa' < \kappa$, for $\sigma \geqq \eta$; then the series is uniformly summable (λ, κ) on any finite stretch of the line $\sigma = \eta$ which does not include any singular point.

THEOREM 43. *If the conditions of the preceding theorem are fulfilled, and the singularities on the line $\sigma = \eta$ are all algebraical infinities of order less than 1, then we may substitute (l, κ) for (λ, κ).*

We do not propose to insert proofs of these theorems†. We may add, however, that the results are capable of considerable generalisation. Thus the nature of the singularities permissible is considerably wider than appears from the enunciations. And in both theorems the hypothesis of regularity on the line $\sigma = \eta$ (except at a finite number of points) is quite unnecessarily restrictive. Thus in Theorem 42 this hypothesis might be replaced by that of continuity for $\sigma \geqq \eta$. In Theorem 43 this would not be sufficient; it would be necessary to impose restrictions similar to those which occur in the theory of the convergence of Fourier's series. The reader will find it instructive to consider the forms of the theorems when $\lambda_n = n$, remembering that summability (l, κ) is then equivalent to convergence (IV, § 4, (3)), and to compare them with the well-known theorems in the theory of Fourier's series to which they are then substantially equivalent.

The differences between Theorems 42 and 43 arise as follows. The formula (3) of § 3 represents the typical mean of the second kind *with its denominator* $w^{-\kappa}$, whereas the corresponding formula of § 2 represents that of the first kind without its denominator. Before studying the convergence of the latter mean the integral which occurs in (2) of § 2 must be divided by ω^{κ}; and it is owing to the presence of this factor that the means of the first kind converge under more general conditions. That the factor occurs in one case and not in the other is in its turn a consequence of the fact that the subject of integration has, for $s = s_0$, an infinity of order $\kappa+1$ in the one case and order unity in the other.

There is another theorem which is also an interesting supplement to Theorem 41.

THEOREM 44. *If the series has a half-plane of absolute convergence, we can replace κ', in the enunciation of Theorem 41, by κ.*

We have $$f(s) = O(|t|^{\kappa})$$
for $\sigma = \eta + \delta$, and $f(s) = O(1)$ for $\sigma = \bar{\sigma} + \delta$, $\bar{\sigma}$ being the abscissa of absolute

* Riesz, 2.

† The proofs depend on a combination of the arguments used in the proof of Theorem 41 with others similar to, but simpler than, those used by Riesz, 5 (pp. 98, 99), in proving and generalising Fatou's theorem (see VI, § 7, Theorem 37).

convergence, and δ any positive number. Hence by Lindelöf's Theorem (Theorem 14) we have

$$f(s) = O(|t|^{\kappa'}),$$

where $\kappa' = (\bar{\sigma} - \eta - \delta)\kappa/(\bar{\sigma} - \eta) < \kappa$, for $\sigma \geqq \eta + 2\delta$. The result now follows from Theorem 41*.

7. From Theorems 38 and 41 we can deduce an important theorem first stated explicitly, for ordinary Dirichlet's series, by Bohr.

Since σ_κ is a decreasing function of κ, the numbers σ_κ tend to a limit, which may be $-\infty$, as $\kappa \to \infty$. We write

$$\lim_{\kappa \to \infty} \sigma_\kappa = S.$$

If S' is any number greater than S, the series is summable (λ, κ), for *some* value of κ, for $\sigma = S'$; and so, by Theorem 38, $f(s)$ is regular and of finite order (III, § 3) for $\sigma > S$. Conversely, if $f(s)$ is regular and of finite order for $\sigma > S'$, it follows from Theorem 41 that the series is summable (λ, κ), for sufficiently large values of κ, for $\sigma > S'$; and so $S' \geqq S$. Hence we deduce

THEOREM 45†. *If η is the least number such that $f(s)$ is regular and of finite order for $\sigma > \eta$, then $\eta = S$.*

8. The following two theorems are in a sense converses of Theorems 42 and 43.

THEOREM 46. *If $\Sigma a_n e^{-\lambda_n s}$ is summable (λ, κ) for $s = s_0$, then*

$$\lim_{\sigma \to \sigma_0} (\sigma - \sigma_0)^{\kappa+1} f(s) = 0$$

uniformly throughout any finite interval of values of t.

THEOREM 47. *If the series is summable (l, κ) for $s = s_0$, we may replace $(\sigma - \sigma_0)^{\kappa+1}$ by $\sigma - \sigma_0$.*

The proofs of these theorems are simple. We indicate that of the first. We may obviously suppose, without loss of generality, that $s_0 = 0$ and $A = 0$.

* This theorem includes a result given by Schnee, **7** (Theorems 3 and 3′). Schnee considers ordinary Dirichlet's series and Cesàro's means of integral order only. See the footnote (‖) to p. 23.

† This theorem was first enunciated in this form by Bohr, **2**. It is however, as shown above, an immediate consequence of Theorem 41 (or Theorem 3 of Riesz's note **1**). See also Bohr, **5**, **6**.

It follows from this theorem, for example, that if the Riemann hypothesis concerning the roots of the ζ-function is true, then the series $\Sigma \mu(n) n^{-s}$ is summable by arithmetic or logarithmic means for $\sigma > \frac{1}{2}$ (Bohr, **2**). As a matter of fact more than this is true: for it has been shown by Littlewood, **2**, that the Riemann hypothesis involves the convergence of the series for $\sigma > \frac{1}{2}$. The best previous result in this direction was due to Landau, **5**, and *H.*, p. 871.

We can then choose ν so that

$$|A^{\kappa}(\tau)| < \epsilon\tau^{\kappa} \qquad (\tau \geqq \nu).$$

By Theorem 24, we have

$$\sigma^{\kappa+1}|f(s)| < \frac{\sigma^{\kappa+1}|s|^{\kappa+1}}{\Gamma(\kappa+1)} \int_0^{\infty} |A^{\kappa}(\tau)| e^{-\sigma\tau} d\tau,$$

and $|s|$ is less than a constant throughout the region under consideration. Hence the preceding expression is less than a constant multiple of

$$\frac{\sigma^{\kappa+1}}{\Gamma(\kappa+1)} \int_0^{\nu} |A^{\kappa}(\tau)| d\tau + \frac{\epsilon\sigma^{\kappa+1}}{\Gamma(\kappa+1)} \int_0^{\infty} \tau^{\kappa} e^{-\sigma\tau} d\tau = \sigma^{\kappa+1} M(\nu) + \epsilon,$$

where $M(\nu)$ depends only on ν; and so is less than 2ϵ when σ is small enough. This proves the theorem: the proof of Theorem 47 is similar, starting from the integral representation of Theorem 29.

From Theorem 47 it follows that the series Σn^{-s} cannot be summable by any arithmetic mean on the line $\sigma = 1$, since the function $\zeta(s)$ has a pole of order 1 at $s = 1$.* On the other hand it follows from Theorem 42, and from the fact that $\zeta(1+ti) = O(\log|t|)$,† that it is summable by any logarithmic mean of positive order at all points of the line save $s = 1$.‡ Compare IV, § 4 (6).

9. Some theorems concerning ordinary Dirichlet's series. All the theorems of this section have been theorems concerning the most general type of Dirichlet's series. We pass now to a few theorems of a more special character. These theorems are valid for forms of λ_n whose rate of increase is sufficiently regular and not too much slower than that of $\log n$: we shall be content to prove them in the simplest and most interesting case, that in which $\lambda_n = \log n$.

THEOREM 48§. *If* $\Sigma a_n n^{-s}$ *is summable* (n, κ)|| *for* $s = s_0$, *it is uniformly summable* (n, κ'), *where* κ' *is the greater of the numbers* $\kappa - \beta$ *and* 0, *in the domain*

$$\sigma \geqq \sigma_0 + \beta, \quad |t| \leqslant T. \P$$

The proof of this theorem is very similar to that of Theorems 23 and 29. We shall consider the case in which $0 < \kappa < 1$. We suppose, as we may do without loss of generality, that $s_0 = 0$ and $A = 0$. We choose a value of β such that $\kappa - \beta \geqq 0$, and we consider the arithmetic

* Landau, *H.*, p. 161.

† Landau, *H.*, p. 169.

‡ For further results relating to the series for $\{\zeta(s)\}^c$ see Riesz, **1**, **2**.

§ For integral orders of summation, and $\sigma > \sigma_0 + \beta$, Bohr, **1**; in the general form, Riesz, **1**.

|| *I.e.* by *arithmetic* means of order κ.

¶ These inequalities might be replaced by $\sigma \geqq \sigma_0 + \beta$, $|\text{am}\, s| \leqq \alpha < \frac{1}{2}\pi$.

mean of order $\kappa-\beta$ at a point s for which $\sigma \geqq \beta$. This mean is easily seen to be (cf. VI, §§ 2 and 5)

$$-w^{-\kappa+\beta}\int_1^w A(u)\frac{d}{du}\{u^{-s}(w-u)^{\kappa-\beta}\}\,du$$

$$=-w^{-\kappa+\beta}\int_1^w A(u)\frac{d}{du}\{(u^{-s}-w^{-s})(w-u)^{\kappa-\beta}\}\,du$$

$$-(\kappa-\beta)\,w^{-\kappa+\beta-s}\int_1^w A(u)(w-u)^{\kappa-\beta-1}\,du.$$

The second term is

$$-w^{-\kappa+\beta-s}A^{\kappa-\beta}(w),$$

and, by Theorem 22, $A^{\kappa-\beta}(w)=o(w^{\kappa})$. Hence this term is of the form $o(1)$, uniformly for $\sigma \geqq \beta$. The first term we integrate by parts, obtaining

$$w^{-\kappa+\beta}\int_1^w A^1(u)\frac{d^2}{du^2}\{(u^{-s}-w^{-s})(w-u)^{\kappa-\beta}\}\,du=J_1+J_2+J_3,$$

say, where J_1, J_2, and J_3 contain under the sign of integration respectively factors

$$s(s+1)\,u^{-s-2}(w-u)^{\kappa-\beta},\quad 2s(\kappa-\beta)\,u^{-s-1}(w-u)^{\kappa-\beta-1},$$

$$(\kappa-\beta)(\kappa-\beta-1)(u^{-s}-w^{-s})(w-u)^{\kappa-\beta-2}.$$

We can now show, by arguments resembling those of VI, § 3 so closely that it is hardly necessary to set them out at length, that J_1 tends to the limit

$$s(s+1)\int_1^\infty A^1(u)\,u^{-s-2}\,du,$$

and J_2 and J_3 to zero, uniformly in the region $\sigma \geqq \beta$, $|t| \leqq T$.

THEOREM 49. *If the series is summable (n,κ), uniformly for $\sigma=\sigma_0$, it is summable (n,κ'), uniformly for $\sigma \geqq \sigma_0+\beta$.*

We apply the argument used in the proof of Theorem 48 to pass from the point σ_0+it to the point $\sigma+it$ with the same ordinate, and take account of the uniformity postulated on the line $\sigma=\sigma_0$. The result then follows substantially as before.

10. By combining Theorems 41 and 48 we arrive at the following theorem.

THEOREM 50. *If, however small δ and ϵ may be, we have*

$$f(s)=O(|t|^{\epsilon})$$

in the half-plane $\sigma \geqq \eta+\delta$, then the series is convergent in every such half-plane, i.e. for $\sigma>\eta$.

For, by Theorem 41, the series is summable (n, ϵ_1), where ϵ_1 is any number greater than ϵ, for $\sigma > \eta + \delta$. Hence, by Theorem 48, it is convergent for $\sigma > \eta + \delta + \epsilon_1$, *i.e.* for $\sigma > \eta$.

Theorem 50 is but a particular case of an important theorem generally known as the 'Schnee-Landau' Theorem.

THEOREM 51. *If* $a_n = O(n^\delta)$ *for all positive values of* δ, *so that the series is absolutely convergent for* $\sigma > 1$, *and*

$$f(s) = O(|t|^k) \qquad (k > 0)$$

uniformly for $\sigma > \eta$, *then the series is convergent for* $\sigma > \zeta$, *where* ζ *is the lesser of the numbers*

$$\frac{\eta + k}{1 + k}, \ \eta + k.*$$

To deal with this theorem and its generalisations would require more space than is at our disposal here, and we must be content to refer to Landau's *Handbuch* and to the original memoirs by Landau and Schnee†. If the second condition is satisfied for all positive values of k, then the series is convergent for $\sigma > \eta$. The first condition then becomes unnecessary, as may be seen at once by applying a linear transformation to the variable s; and so we obtain Theorem 50.

11. We can obtain another important theorem by combining Theorem 49 with the result proved at the end of § 5. Suppose that $f(s)$ is bounded in every half-plane $\sigma \geqq \eta + \delta > \eta$. Then, if δ and ϵ are chosen arbitrarily, the series is uniformly summable (n, ϵ) for $\sigma \geqq \eta + \delta$, and therefore, by Theorem 49, uniformly convergent for $\sigma \geqq \eta + \delta + \epsilon$. We thus obtain

THEOREM 52. *If* $f(s)$ *is bounded in every half-plane* $\sigma \geqq \eta + \delta > \eta$, *then the series is uniformly convergent in every such half-plane.*

This theorem was first given by Bohr‡. Its converse is obviously trivial.

Before leaving these theorems we may make a few additional remarks. Theorems 48—52 may be extended§ to any type of series $\Sigma a_n e^{-\lambda_n s} = \Sigma a_n l_n^{-s}$ for which a positive constant g exists such that

$$\frac{l_{n+1}}{l_{n+1} - l_n} = O(l_n^g) \qquad \text{(1)}.$$

* The first number gives the better result if $\eta + k > 0$, the second if $\eta + k < 0$.

† Landau, *H.*, pp. 853 *et seq.* See also Landau, **5**, **7**; Schnee, **4**, **7**; Bohr, **2**, **5**, **10**.

‡ Bohr, **3**, **8**.

§ See the memoirs cited in footnote (†).

This hypothesis ensures that the increase of l_n is *not too slow*; it is satisfied, for instance, if $l_n = n$ or $l_n = e^n$, but not if $l_n = \log n$. It is easy to show that the condition (1) is equivalent to either of the following:

$$l_n^{-h} = O(l_{n+1} - l_n) \quad \text{.............................(2)},$$

$$e^{-k\lambda_n} = O(\lambda_{n+1} - \lambda_n) \quad \text{.............................(3)},$$

where h and k are positive*.

Considerations of space forbid us from giving details of these generalisations. We would only warn the reader that the proofs, involving as they do in some places an appeal to the delicate Theorem 22, are not entirely simple, especially when the increase of l_n is very rapid and irregular.

The line $\sigma = \eta$ such that the series is uniformly convergent for $\sigma \geqq \eta + \delta$, but not for $\sigma \geqq \eta - \delta$, however small be δ, has been called by Bohr the *line of uniform convergence*. It has been shown by Bohr† that, when the numbers λ_n are linearly independent, the line of uniform convergence is identical with the line of absolute convergence: but he has given an example of a series (naturally corresponding to a non-independent sequence of λ's) which possesses a half-plane of uniform convergence and no half-plane of absolute convergence.

12. Convexity of the abscissa σ_κ, considered as a function of κ. It was shown by Bohr ‡ that the abscissa of summability σ_r, of integral order, belonging to an ordinary Dirichlet's series, satisfies the inequalities

$$\sigma_{r+1} \leqq \sigma_r \leqq \sigma_{r+1} + 1 \quad \text{.......................(1)},$$

$$\sigma_r - \sigma_{r+1} \geqq \sigma_{r+1} - \sigma_{r+2} \quad \text{.....................(2)}.$$

Of the inequalities (1), the first is an obvious corollary of Theorem 16 (cf. VI, § 4); and the second is an obvious corollary of Theorem 48. The inequalities (2) lie deeper.

The property which is expressed by the inequalities (2) was then considered by Hardy and Littlewood §, who proved more precise theorems of which Bohr's inequalities are corollaries. Their results have since been extended by Riesz ||, so as to apply to the most general type of Dirichlet's series and to all orders of summation integral or non-integral. In particular it has been proved that *the abscissa σ_κ is in all cases a convex function of κ.*

It was also shown by Bohr ¶ that the conditions (1) and (2) are

* These conditions are rather wider than that adopted by Schnee and Landau, and are substantially the same as that adopted by Bohr. It is natural to suppose h and k positive, but not necessary; for if (3), for example, is satisfied with $k \leq 0$, it is plainly satisfied with $k > 0$.

† Bohr, **7**: cf. III, § 7. ‡ Bohr, **2**, **5**. § Hardy and Littlewood, **2**.

|| In a memoir as yet unpublished. ¶ Bohr, **5**.

necessary and sufficient that a given sequence σ_r should be the abscissae of summability of *some* ordinary Dirichlet's series*.

13. Summation of Dirichlet's Series by other methods. It is natural to enquire whether methods of summation different in principle from those which we have considered may not be useful in the theory. The first to suggest itself is Borel's exponential method. The application of this method to ordinary Dirichlet's series has been considered by Hardy and by Fekete†. It has been shown, for example, that the regions of summability, and of absolute summability, are half-planes; and that the method at once gives the analytical continuation all over the plane of certain interesting classes of series. But the method is not one which seems likely to render great services to the general theory.

Riesz‡ has considered methods of summation related to Borel's, and its generalisation by Mittag-Leffler, somewhat as the typical means of this section are related to Cesàro's original means. These methods lead to representations of the function associated with the series which differ fundamentally in one very important respect from those afforded by the theory of typical means. Their domains of application may, like Borel's polygon of summability, or Mittag-Leffler's *étoile*, be defined simply by means of the singular points of the function, and necessarily contain singular points on their frontier.

VIII

THE MULTIPLICATION OF DIRICHLET'S SERIES

1. We shall be occupied in this section with the study of a special problem, interesting on account of the variety and elegance of the results to which it has led, and important on account of its applications in the Analytic Theory of Numbers§.

* The construction given by Bohr (*l.c.* pp. 127 *et seq.*), for a series with given abscissae may be simplified by using the series $\Sigma e^{Ain^a} n^{-s}$ of IV, § 4, (5) as a 'simple element' in place of the series which he uses.

† Hardy, **3**; Fekete, **1**.

‡ Riesz, **6**.

§ In this connection we refer particularly to Landau, **4**, and *H.*, pp. 750 *et seq.*

We denote by A and B the series

$$a_1 + a_2 + \dots, \quad b_1 + b_2 + \dots,$$

and by C the 'product-series'

$$c_1 + c_2 + \dots,$$

where c_n is a function of the a's and b's, to be defined more precisely in a moment. We shall also use A, B, C to denote the *sums* of the series, when they are convergent or summable.

When C is formed in accordance with Cauchy's rule*, we have

$$c_p = a_1 b_p + a_2 b_{p-1} + \dots + a_p b_1 = \sum_{m+n=p+1} a_m b_n.$$

Cauchy's rule for multiplication is, however, only one among an infinity. We are led to it by arranging the formal product of the power series $\Sigma a_m x^m$, $\Sigma b_n x^n$ in powers of x and putting $x = 1$, or, what is the same thing, by arranging the formal product of the Dirichlet's series

$$\Sigma a_m e^{-ms}, \quad \Sigma b_n e^{-ns}$$

according to the ascending order of the sums $m + n$, associating together all the terms for which $m + n$ has the same value, and then putting $s = 0$. It is clear that we arrive at a generalisation of our conception of multiplication by considering the general Dirichlet's series

$$\Sigma a_m e^{-\lambda_m s}, \quad \Sigma b_n e^{-\mu_n s}$$

and arranging their formal product according to the ascending order of the sums $\lambda_m + \mu_n$. Let (ν_p) be the ascending sequence formed by all the values of $\lambda_m + \mu_n$†. Then the series $C = \Sigma c_p$, where

$$c_p = \sum_{\lambda_m + \mu_n = \nu_p} a_m b_n,$$

will be called *the Dirichlet's product of the series* A, B, *of type* (λ, μ).

Thus if $\lambda_m = \log m$, $\mu_n = \log n$, so that we are dealing with ordinary Dirichlet's series, then $\nu_p = \log p$ and

$$c_p = \sum_{mn=p} a_m b_n = \sum_d a_d b_{p/d},$$

the latter summation extending to all the divisors d of p.

* See *e.g.* Bromwich, *Infinite series*, p. 83.

† It is generally the case in applications that the λ and μ sequences are the same. Any case can be formally reduced to this case by regarding all the numbers λ_m and μ_n as forming one sequence and attributing to each series a number of terms with zero coefficients (Landau, *H.*, p. 750). In the most important cases (*e.g.* $\lambda_m = m$, $\lambda_m = \log m$) the ν sequence is also the same, but of course this is not generally true. In the theoretically general case no two values of $\lambda_m + \mu_n$ will be equal.

2. The three classical theorems relating to ordinary multiplication (Cauchy's, Mertens', and Abel's) have their analogues in the general theory.

THEOREM 53. *If A and B are absolutely convergent, then C is absolutely convergent and $AB = C$.*

This theorem is merely a special case of the classical theorem which asserts that the absolutely convergent double series $\Sigma\, a_m b_n$ may be summed indifferently in any manner we please*.

THEOREM 54. *If A is absolutely convergent and B convergent, then C is convergent and $AB = C$.*†

We shall prove that $\Sigma\, a_m b_n$ converges to the sum AB when arranged as a simple series so that $a_m b_n$ comes before $a_{m'} b_{n'}$ if $\lambda_m + \mu_n < \lambda_{m'} + \mu_{n'}$ (the order of the terms for which $\lambda_m + \mu_n$ has the same value being indifferent). Theorem 54 then follows by bracketing all the terms for which $\lambda_m + \mu_n$ has the same value.

Suppose first that $B = 0$. Let S_ν be any partial sum of the new series, and let a_k be the a of highest rank that occurs in it. Then

$$S_\nu = \sum_{p=1}^{k} a_p B_r,$$

where r is a function of k and p. Suppose that S_ν contains a term $a_\gamma b_\gamma$. Then it contains *all* the terms

$$a_p b_q \qquad (p \leqq \gamma,\ q \leqq \gamma).\ddagger$$

Thus $k \geqq \gamma$ and $r \geqq \gamma$ for $p = 1, 2, \dots \gamma$.

Now we can choose γ so that

$$|B_r| < \epsilon \qquad (r \geqq \gamma),$$

and

$$\sum_{\gamma+1}^{\infty} |a_p| < \epsilon.$$

Then

$$|S_\nu| < \epsilon \sum_{1}^{\gamma} |a_p| + M \sum_{\gamma+1}^{k} |a_p| < \epsilon(\mathbf{A} + M),$$

where $\mathbf{A}$ denotes the sum of the series $\Sigma\,|a_p|$ and M is any number greater than the greatest value of $|B_r|$. Thus $S_\nu \to 0$ as $\gamma \to \infty$, that is to say as $k \to \infty$.

* See *e.g.* Bromwich, *Infinite series*, p. 81. This theorem is not merely a special case of Theorem 54, because it asserts the *absolute* convergence of the product series.

† Stieltjes, **2**; Landau, **4**, and *H.*, p. 752. See also Wigert, **1**.

‡ This is the kernel of the proof. The reader will find that a figure will help to elucidate the argument.

Secondly, suppose $B \neq 0$. We form a new series B' for which

$$b_1' = b_1 - B, \quad b_2' = b_2, \quad b_3' = b_3, \ldots.$$

Then, by what precedes, $\Sigma a_m b_n'$ converges to zero, and so $\Sigma a_m b_n$ converges to AB.

3. THEOREM 55. *If the series A, B, C are all convergent, then $AB = C$.*

This is the analogue of Abel's theorem for power series*. We shall deduce it from a more general theorem, the analogue for Dirichlet's series of a well-known theorem of Cesàro†.

THEOREM 56. *If A is summable (λ, α) and B is summable (μ, β), then C is summable $(\nu, \alpha + \beta + 1)$, and $AB = C$.*

If $\gamma = \alpha + \beta + 1$ we have

$$A_\lambda^\alpha(\omega) = \Sigma\, a_m (\omega - \lambda_m)^\alpha, \quad B_\mu^\beta(\omega) = \Sigma\, b_n (\omega - \mu_n)^\beta, \quad C_\nu^\gamma(\omega) = \Sigma\, c_p (\omega - \nu_p)^\gamma,$$

the summations being limited respectively by the inequalities $\lambda_m < \omega$, $\mu_n < \omega$, $\nu_p < \omega$. Then

$$C_\nu^\gamma(\omega) = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+1)\,\Gamma(\beta+1)} \int_0^\omega A_\lambda^\alpha(\tau)\, B_\mu^\beta(\omega - \tau)\, d\tau \quad \ldots\ldots(1).$$

For consider the term $a_m b_n$. It occurs in $C_\nu^\gamma(\omega)$ if $\lambda_m + \mu_n < \omega$, and its coefficient is

$$(\omega - \lambda_m - \mu_n)^\gamma.$$

The term a_m occurs in $A_\lambda^\alpha(\tau)$ if $\lambda_m < \tau$, with coefficient $(\tau - \lambda_m)^\alpha$, and b_n occurs in $B_\mu^\beta(\omega - \tau)$ if $\mu_n < \omega - \tau$, with coefficient $(\omega - \tau - \mu_n)^\beta$. Hence $a_m b_n$ occurs on the right-hand side of (1) if $\lambda_m + \mu_n < \omega$, and its coefficient is

$$\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+1)\,\Gamma(\beta+1)} \int_{\lambda_m}^{\omega - \mu_n} (\tau - \lambda_m)^\alpha (\omega - \tau - \mu_n)^\beta\, d\tau = (\omega - \lambda_m - \mu_n)^\gamma.$$

* The theorem was first given by Landau, **4**, in the case in which at least one of the series $\Sigma\, a_m e^{-\lambda_m s}$, $\Sigma\, b_n e^{-\mu_n s}$ possesses a region of absolute convergence. His proof depended on considerations of function-theory. A purely arithmetic and completely general proof was then discovered independently by Phragmén, Riesz, and Bohr. This proof depends on the particular case of Theorem 56 in which $\alpha = \beta = 0$. See Landau, *H.*, pp. 762, 904; Riesz, **2**; Bohr, **2**.

† Cesàro, **1**; see also Bromwich, *Infinite series*, p. 316. Cesàro and Bromwich consider only integral orders of summability. The extension to non-integral orders is due to Knopp, **2**, and Chapman, **1**.

Thus (1) is established. But

$$A_{\lambda}^{\alpha}(\tau) \sim A\tau^{\alpha}, \quad B_{\mu}^{\beta}(\tau) \sim B\tau^{\beta};$$

and therefore, by Lemma 5,

$$C_{\nu}^{\gamma}(\omega) \sim AB\,\omega^{\gamma}.$$

This proves the theorem. In particular, if A and B are convergent, the product series is summable $(\nu, 1)$. Theorem 55 then follows from Theorems 56 and 16.

4. The following generalisation of Theorem 54 provides an interesting companion theorem to Theorem 56.

THEOREM 57. *If A is absolutely convergent, and B is summable (μ, β), then C is summable (ν, β) and $C=AB$.**

In this theorem the λ-sequence is at our disposal. It is evidently enough (cf. § 2) to prove the theorem in the particular case when $B=0$.

We have

$$C_{\nu}^{\beta}(\omega) = \sum_{\lambda_m+\mu_n<\omega} a_m b_n (\omega-\lambda_m-\mu_n)^{\beta} = \sum_{\lambda_m<\omega} a_m B_{\mu}^{\beta}(\omega-\lambda_m).$$

There is a constant M such that

$$|B_{\mu}^{\beta}(\tau)| < M\tau^{\beta}$$

for all values of τ; and we can choose ω so that (1) the M on the right-hand side of this inequality can be replaced by ϵ if $\tau_m \geqq \frac{1}{2}\omega$, and (2)

$$\sum_{\frac{1}{2}\omega \leqq \lambda_m < \omega} |a_m| < \epsilon.$$

Then we have

$$|C_{\nu}^{\beta}(\omega)| < M \sum_{\frac{1}{2}\omega \leqq \lambda_m < \omega} |a_m|(\omega-\lambda_m)^{\beta} + \epsilon \sum_{\lambda_m<\omega} |a_m|(\omega-\lambda_m)^{\beta},$$

$$|\omega^{-\beta} C_{\nu}^{\beta}(\omega)| < M \sum_{\frac{1}{2}\omega \leqq \lambda_m < \omega} |a_m| + \epsilon\mathbf{A} < \epsilon(M+\mathbf{A}),$$

and so

$$\omega^{-\beta} C_{\nu}^{\beta}(\omega) \to 0:$$

which proves the theorem.

5. The next theorem which we shall state is one whose general idea is analogous to those of the 'Tauberian' theorems of VI, § 7, and in particular Theorem 35. We therefore omit the proof.

* For the special case of multiplication in accordance with Cauchy's rule, see Hardy and Littlewood, **2** (Theorem 35), where further theorems on the multiplication of series will be found. The particular theorem proved there is however a special case of one given previously by Fekete, **3**.

THEOREM 58. *If*

$$a_m = O\left(\frac{\lambda_m - \lambda_{m-1}}{\lambda_m}\right), \quad b_n = O\left(\frac{\mu_n - \mu_{n-1}}{\mu_n}\right),$$

*then the convergence of A and B is enough to ensure that of C.**

6. Our last two theorems are of a different character.

THEOREM 59†. *If*

(1) $\tau \geqq 0, \ \tau' \geqq 0, \ \tau + \tau' > 0, \ \rho + \tau \geqq \rho', \ \rho' + \tau' \geqq \rho,$

(2) *the series* $\Sigma a_m e^{-\lambda_m s}$ *is convergent for* $s = \rho$, *and absolutely convergent for* $s = \rho + \tau$,

(3) *the series* $\Sigma b_n e^{-\mu_n s}$ *is convergent for* $s = \rho'$, *and absolutely convergent for* $s = \rho' + \tau'$;

then the series $\Sigma c_p e^{-\nu_p s}$ *is convergent for*

$$s = \frac{\rho\tau' + \rho'\tau + \tau\tau'}{\tau + \tau'}.$$

We shall give a proof of this theorem only in the simplest and most interesting case, viz. that in which

$$\lambda_m = \log m, \quad \mu_n = \log n, \quad \nu_p = \log p,$$

so that the series are ordinary Dirichlet's series, and

$$\rho = \rho' = 0.$$

We can then suppose that τ and τ' are any numbers greater than 1, so that

$$\frac{\rho\tau' + \rho'\tau + \tau\tau'}{\tau + \tau'}$$

may be any number greater than $\frac{1}{2}$. The theorem therefore asserts that, if A and B are convergent, and

$$c_p = \sum_{mn=p} a_m b_n,$$

then $\Sigma c_p p^{-s}$ is convergent for $\sigma > \frac{1}{2}$. In this case, however, it is possible to prove rather more.

* This theorem is not a corollary of Theorem 35. The conditions do not ensure that $c_p = O\{(\nu_p - \nu_{p-1})/\nu_p\}$. The theorem was proved, in the particular case $\lambda_m = m$, $\mu_n = n$, by Hardy, **2**; and in the general case by Hardy, **7**. Hardy however supposed the indices λ_m, μ_n subject to the conditions

$$\lambda_m - \lambda_{m-1} = o(\lambda_m), \quad \mu_n - \mu_{n-1} = o(\mu_n).$$

That these conditions are unnecessary was shown by Rosenblatt, **2**.

† Landau, **4**, and *H.*, p. 755.

THEOREM 60. *If A and B are convergent, then* $\Sigma \frac{c_p}{\sqrt{p}}$ *is convergent**.

We shall prove, in fact, that $\Sigma c_p p^{-s}$ is uniformly convergent along any finite stretch of the line $s=\frac{1}{2}+ti$.

Let us write $$A_x = \sum_{m>x} a_m, \quad A = A(x) + A_x,$$

and similarly for B. We have

$$\sum_m^\infty a_\nu \nu^{-s} = \sum_m^\infty (A_{\nu-1} - A_\nu)\nu^{-s}$$
$$= A_{m-1} m^{-s} + \sum_m^\infty A_\nu \{\nu^{-s} - (\nu+1)^{-s}\} = o\left(\frac{1}{\sqrt{m}}\right),$$

as $$m^{-s} = O\left(\frac{1}{\sqrt{m}}\right), \quad \sum_m^\infty |\nu^{-s} - (\nu+1)^{-s}| = O\left(\frac{1}{\sqrt{m}}\right), \quad A_m = o(1).$$

Similarly $$\sum_n^\infty b_\nu \nu^{-s} = o\left(\frac{1}{\sqrt{n}}\right);$$

and these relations all hold uniformly as regards t. We observe now that

$$\sum_1^x c_p p^{-s}$$

includes all products of pairs of terms $a_m m^{-s}$, $b_n n^{-s}$ for which $mn \leqq [x]$, and

$$\sum_1^{\sqrt{x}} a_m m^{-s} \times \sum_1^{\sqrt{x}} b_n n^{-s}$$

all for which $m \leqq \sqrt{x}$, $n \leqq \sqrt{x}$; and that, if $mn \leqq [x]$, one at least of m and n is not greater than $\sqrt{x}$. It follows that

$$\sum_1^x c_p p^{-s} - \sum_1^{\sqrt{x}} a_m m^{-s} \sum_1^{\sqrt{x}} b_n n^{-s}$$
$$= \sum_1^{\sqrt{x}} a_m m^{-s} \sum_{\sqrt{x}}^{x/m} b_\nu \nu^{-s} + \sum_1^{\sqrt{x}} b_n n^{-s} \sum_{\sqrt{x}}^{x/n} a_\nu \nu^{-s}$$
$$= o(x^{-\frac{1}{4}}) \sum_1^{\sqrt{x}} \frac{1}{\sqrt{m}} + o(x^{-\frac{1}{4}}) \sum_1^{\sqrt{x}} \frac{1}{\sqrt{n}} = o(1);\dagger$$

which proves the theorem.

It was suggested by Cahen‡ that the convergence of A and B should involve the convergence of $\Sigma c_p p^{-s}$ for $\sigma > 0$, and not merely for $\sigma \geqq \frac{1}{2}$ (as is shown by Theorem 60). This question, the answer to which remained for long doubtful, was ultimately decided by Landau§, who showed by an example that Cahen's hypothesis was untrue.

* Stieltjes, **1**, **2**. See also Landau, **4**, and *H.*, pp. 759 *et seq.*

† Since $$\sum_{\sqrt{x}}^{x/m} b_\nu \nu^{-s} = o(x^{-\frac{1}{4}}) - o\left(\sqrt{\frac{m}{x}}\right) = o(x^{-\frac{1}{4}});$$

and similarly $$\sum_{\sqrt{x}}^{x/n} a_\nu \nu^{-s} = o(x^{-\frac{1}{4}}).$$

‡ Cahen, **1**. § Landau, **5**, and *H.*, p. 773.

This may be seen very simply by means of Bohr's example (III, § 7) of a function $f(s)$, convergent for $\sigma>0$, for which $\mu(\sigma)=1-\sigma$ for $0<\sigma<1$. If we square this function, we obtain a function for which $\mu(\sigma)=2-2\sigma$ for $0<\sigma<1$, so that $\mu(\sigma)>1$ if $\sigma<\frac{1}{2}$. It follows from Theorem 12 that the squared series cannot converge for $\sigma<\frac{1}{2}$, and hence that the number $\frac{1}{2}$ which occurs in Theorem 60 cannot possibly be replaced by any smaller number*.

* Bohr, **5**.

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H. Bohr

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(2) 'Über die Summabilität Dirichletscher Reihen', *Göttinger Nachrichten*, 1909, pp. 247–262.

(3) 'Sur la convergence des séries de Dirichlet', *Comptes Rendus*, 1 Aug. 1910.

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(1) 'Sur la fonction $\zeta(s)$ de Riemann et sur des fonctions analogues', *Ann. de l'École Normale*, ser. 3, vol. 11, 1894, pp. 75–164.

E. Cesàro

(1) 'Sur la multiplication des séries', *Bulletin des Sciences Math.*, vol. 14, 1890, pp. 114–120.

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